Rationality of Three-Dimensional Quotients by Monomial Actions

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Abstract. Let G be a finite 2-group and K be a field satisfying that (i) char $K \neq 2$, and (ii) $\sqrt{a} \in K$ for any $a \in K$. If G acts on the rational function field $K(x,y,z)$ by monomial K-automorphisms, then the fixed field $K(x,y,z)^G$ is rational (= purely transcendental) over K. Applications of this theorem will be given.

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§1. Introduction

Let K be any field and $K(x_1,\ldots,x_n)$ be the rational function field of n variables over K. A K-automorphism σ of $K(x_1,\ldots,x_n)$ is said to be a monomial automorphism if

$$\sigma(x_j)=a_j(\sigma)\prod_{i=1}^{n}x_i^{m_{ij}}$$

where $(m_{ij})_{1\le i,j\le n}\in GL_n(\mathbb{Z})$ and $a_j(\sigma)\in K\backslash\{0\}$. If $a_j(\sigma)=1$ for all $1\le j\le n$, then σ is called a purely monomial automorphism. A group action on $K(x_1,\ldots,x_n)$ by monomial K-automorphisms is also called a multiplicative group action. Multiplicative actions are crucial in solving rationality problems for linear group actions, e.g. Noether problem (see, for examples, [Sw; CHKP]).

However, the rationality problem of a multiplicative action on $K(x_1,\ldots,x_n)$ is rather intricate. We recall some previously known results first.

Theorem 1.1 (Hajja [Ha1; Ha2])**.** *Let K be any field and $K(x,y)$ be the rational function field of two variables over K. If G is a finite group acting on the rational function field $K(x,y)$ by monomial K-automorphisms, then the fixed field $K(x,y)^G$ is rational* (*= purely transcendental*) *over K.*

Theorem 1.2 (Hajja, Kang, Hoshi and Rikuna [HK1; HK2; HR])**.** *Let K be any field and $K(x,y,z)$ be the rational function field of three variables over K. If G is a finite group acting on $K(x,y,z)$ by purely monomial K-automorphisms, then the fixed field $K(x,y,z)^G$ is rational over K.*

In the above Theorem 1.2, it is impossible to replace the assumption "purely monomial K-automorphisms" by the weaker assumption "monomial K-automorphisms" as the fixed field $K(x,y,z)^{\langle\sigma\rangle}$ is not rational over K where the monomial K-automorphism σ is defined by $\sigma: x\mapsto y\mapsto z\mapsto -1/(xyz)$ (see the last paragraph of [Ha1]). However, if we assume that K is an algebraically closed field, then we can get an affirmative result for monomial group actions. In fact, the main result of this article is the following theorem.

Theorem 1.3. *Let G be a finite 2-group and K be a field satisfying that*

(i) $\operatorname{char} K\neq 2$*, and*

(ii) $\sqrt{a}\in K$ *for any* $a\in K$.

If G acts on the rational function field $K(x,y,z)$ by monomial K-automorphisms, then the fixed field $K(x,y,z)^G$ is rational over K.

In the above theorem, it is assumed that $\operatorname{char} K\neq 2$. We may as well show that $K(x,y,z)^G$ is rational over K when $\operatorname{char} K=2$ and K satisfies the assumption that $\sqrt{a}\in K$ for any $a\in K$, by using the techniques in [HK1; HK2] and the method

developed in Section 3 of this paper. Because we intend to highlight the main techniques in the proof of Theorem 1.3 and because we try to shorten the length of this article, we omit the proof of the case when $\operatorname{char} K = 2$.

We will give another remark on Theorem 1.3. It is impossible to generalize Theorem 1.3 to the case of rational function fields of four variables. In fact, if $G \simeq C_2^3$, there is a monomial action of G on $\mathbb{C}(x_1, x_2, x_3, x_4)$ so that the fixed field $\mathbb{C}(x_1, x_2, x_3, x_4)^G$ is not retract rational over $\mathbb{C}$; in particular, it is not rational over $\mathbb{C}$ [CHKK, Example 5.11].

An application of Theorem 1.3 is the following theorem.

Theorem 1.4. *Let G be a finite 2-group and K be a field satisfying that*

(i) $\operatorname{char} K \neq 2$*, and*

(ii) $\sqrt{a} \in K$ *for any* $a \in K$*.*

Suppose that $\rho : G \to GL(V)$ is a linear representation of G over K such that either $\dim_K V \leq 5$ or ρ is the direct sum of three 2-dimensional irreducible representations. Then both the fixed field $K(V)^G$ and the quotient $P(V)/G$ are rational over K.

We remark that, if we assume that the field K is algebraically closed, the conclusion of the above Theorem 1.3 is still valid for any finite group G, i.e. no assumption about G being a 2-groups is necessary. We refer the reader to [Pr, Section 5] for some key ideas of the proof. We choose to publish parts of the general result because of two reasons. First of all, the proof of the general case is rather long and complicated; we had better publish this result in two separate papers. Second, the result for 2-groups will be used in a forthcoming paper on Noether's problem and Bogomolov multipliers (i.e. the unramified Brauer groups) for groups of order 64 [CHKK].

We will point out that Theorem 1.4 provides a quick proof for the special case when K is algebraically closed of the following theorem.

Theorem 1.5 (Chu, Hu, Kang and Prokhorov [CHKP])**.** *Let G be a group of order 32 with exponent e, and K be a field satisfying*
(i) $\operatorname{char} K = 2$*, or*
(ii) $\operatorname{char} K \neq 2$ *and K contains a primitive e-th root of unity,*
Then $K(x_g : g \in G)^G$ is rational over K.

Note that Theorem 1.5 was proved using the classification of groups of order 32, i.e. the structures and representations of these groups provided by the data base of GAP [CHKP]. If we choose to avoid using the classification of groups of order 32, we still have a proof, thanks to Theorem 1.4, but we need the stronger assumptions that $\operatorname{char} K \neq 2$ and $\sqrt{a} \in K$ for any $a \in K$ (e.g. K is algebraically closed).

We will organize this paper as follows. In Section 2 we recall several preliminary results which will be used subsequently. Section 3 contains the complete proof of Theorem 1.3; the proof uses only elementary methods and will be more accessible to most readers. Another proof of Theorem 1.3 for the case when the ground field K is

algebraically closed and $\operatorname{char} K = 0$ will be given in Section 4. This new proof is shorter than that in Section 3, more conceptual, and no classification of finite subgroups of $GL_3(\mathbb{Z})$ [Ta] is required; but the price is that some machinery in algebraic geometry is used. The proof of Theorem 1.4 will be given in Section 5. As an application of it, we will give a quick proof of Theorem 1.5 provided that K is algebraically closed.

Standing Notations. Throughout this paper, K is a field, $K(x_1, \ldots, x_n)$ or $K(x, y, z)$ is the rational function field over K. We will denote by ζ_n a primitive n-th root of unity. Whenever we write $\zeta_n \in K$, it is understood that either $\operatorname{char} K = 0$ or $\operatorname{char} K > 0$ with $\operatorname{char} K \nmid n$.

If G is a finite group, the exponent of G is $\operatorname{lcm}\{\operatorname{ord}(g) : g \in G\}$ where $\operatorname{ord}(g)$ is the order of an element $g \in G$. The cyclic group of order n and the dihedral group of order $2n$ will be denoted by C_n and D_n respectively.

If G is a finite group acting on $K(x_1, \ldots, x_n)$ by K-automorphism, the actions of G are called monomial actions if for any $\sigma \in G$, any $1 \le j \le n$, $\sigma \cdot x_j = a_j(\sigma) \cdot \prod_{1 \le i \le n} x_i^{m_{ij}}$ where $m_{ij} \in \mathbb{Z}$ and $a_j(\sigma) \in K \backslash \{0\}$. The actions are called purely monomial actions if they are monomial actions satisfying $a_j(\sigma) = 1$ for any $\sigma \in G$, any $1 \le j \le n$. In the rational function field $K(x_1, \ldots, x_n)$, an element of the form $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ where each $\lambda_i \in \mathbb{Z}$ will be called a power product in $x_1, \ldots, x_n$; we refrain from calling it a monomial to avoid possible confusion with the monomial action and with $a \cdot x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ where $a \in K$.

§2. Preliminaries

In this section we recall several results which will be used in the proof of Theorem 1.3.

Theorem 2.1 ([HK3, Theorem 1])**.** *Let G be a finite group acting on $L(x_1, \ldots, x_n)$, the rational function field of n variables over a field L. Suppose that*

(i) *for any $\sigma \in G$, $\sigma(L) \subset L$,*

(ii) *the restriction of the action of G to L is faithful, and*

(iii) *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \sigma(x_2) \\ \vdots \\ \sigma(x_n) \end{pmatrix} = A(\sigma) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + B(\sigma)$$

where $A(\sigma) \in GL_n(L)$ and $B(\sigma)$ is an $n \times 1$ matrix over L.

Then there exist elements $z_1, \ldots, z_n \in L(x_1, \ldots, x_n)$ which are algebraically independent over L and $L(x_1, \ldots, x_n) = L(z_1, \ldots, z_n)$ so that $\sigma(z_i) = z_i$ for any $\sigma \in G$, any $1 \le i \le n$.

Theorem 2.2 ([AHK, Theorem 3.1])**.** *Let L be any field, $L(x)$ the rational function field of one variable over L, and G a finite group acting on $L(x)$. Suppose that, for any $\sigma \in G$, $\sigma(L) \subset L$ and $\sigma(x) = a_\sigma \cdot x + b_\sigma$ where $a_\sigma, b_\sigma \in L$ and $a_\sigma \neq 0$. Then $L(x)^G = L^G(f)$ for some polynomial $f \in L[x]$. In fact, if $m = \min\{\deg g(x) : g(x) \in L[x]^G \backslash L^G\}$, any polynomial $f \in L[x]^G$ with $\deg f = m$ satisfies the property $L(x)^G = L^G(f)$.*

Theorem 2.3 ([CHK, Theorem 2.3])**.** *Let K be any field, $a, b \in K\backslash\{0\}$ and $\sigma : K(x,y) \to K(x,y)$ be a K-automorphism of the rational function field $K(x,y)$ defined by $\sigma(x) = a/x$, $\sigma(y) = b/y$. Then $K(x,y)^{\langle\sigma\rangle} = K(u,v)$ where*

$$u = \frac{x - \frac{a}{x}}{xy - \frac{ab}{xy}}, \quad v = \frac{y - \frac{b}{y}}{xy - \frac{ab}{xy}}.$$

Moreover,

$$\begin{aligned} x + (a/x) &= (-bu^2 + av^2 + 1)/v, \\ y + (b/y) &= (bu^2 - av^2 + 1)/u, \\ xy + (ab/(xy)) &= (-bu^2 - av^2 + 1)/(uv), \end{aligned}$$

and

$$\frac{x - \frac{a}{x}}{\frac{bx}{y} - \frac{ay}{x}} = \frac{u}{bu^2 - av^2}. \tag{2.1}$$

Proof. The proof of Formula (2.1) was given in [CHK, p.156]. □

Theorem 2.4 ([Ka1, Theorem 2.4])**.** *Let K be any field. Define a K-automorphism σ on the rational function field $K(x,y)$ by $\sigma(x) = a/x$, $\sigma(y) = \{b_1[x + (a/x)] + b_2\}/y$ where $a, b_1, b_2 \in K$ with $ab_1 \neq 0$. Then $K(x,y)^{\langle\sigma\rangle} = K(u,v)$ where*

$$u = \frac{x - \frac{a}{x}}{xy - \frac{ab}{xy}}, \quad v = \frac{y - \frac{b}{y}}{xy - \frac{ab}{xy}}.$$

with $b = b_1[x + (a/x)] + b_2$.

Theorem 2.5 ([HKO, Theorem 6.7; HK2, Theorem 2.4])**.** *Let F be a field with* $\operatorname{char} F \neq 2$*, $E = F(\alpha)$ be a field extension of F defined by $\alpha^2 = a \in F\backslash\{0\}$. Let $E(x,y)$ be the rational function field of two variables over E and σ be an F-automorphism on $E(x,y)$ defined by*

$$\sigma(\alpha) = -\alpha, \quad \sigma(x) = x, \quad \sigma(y) = b(x^2 - c)/y$$

where $b, c \in F$ with $b \neq 0$. Then $E(x,y)^{\langle\sigma\rangle}$ is rational over F if and only if the Hilbert symbol $(a,b)_2$ is trivial in the field $F(\sqrt{ac})$.

Theorem 2.6 ([Ka4, Theorem 1.4]). *Let K be a field and G be a finite group. Assume that*

(i) *G contains an abelian normal subgroup H so that G/H is cyclic of order n;*

(ii) *$\mathbb{Z}[\zeta_n]$ is a unique factorization domain; and*

(iii) *$\zeta_e \in K$ where e is the exponent of G.*

If $G \to GL(V)$ is any finite-dimensional linear representation of G over K, then $K(V)^G$ is rational over K.

Definition 2.7. *Let G be a finite group acting on $K(x_1, \ldots, x_n)$ by monomial K-automorphisms. Let $z_1, z_2, \ldots, z_n \in K(x_1, \ldots, x_n)$ satisfy*

(i) $\operatorname{trdeg}_K K(z_1, \ldots, z_n) = n$,

(ii) *for any $\sigma \in G$, any $1 \le j \le n$, $\sigma \cdot z_j = c_j(\sigma) \cdot \prod_{1 \le i \le n} z_i^{m_{ij}}$ where $m_{ij} \in \mathbb{Z}$ and $c_j(\sigma) \in K \backslash \{0\}$.*

We will define a group homomorphism $\rho_{\underline{z}} : G \to GL_n(\mathbb{Z})$ by $\rho_{\underline{z}}(\sigma) = (m_{ij})_{1 \le i,j \le n}$ if $\sigma \cdot z_j$ is given by (ii). *Note that $\underline{z}$ denotes the ordered transcendental basis $(z_1, \ldots, z_n)$; if $\underline{z}$ is understood from the context, we will simply write ρ for $\rho_{\underline{z}}$.*

Lemma 2.8. *Let K be any field and G be a finite group acting on $K(x_1, \ldots, x_n)$ by monomial K-automorphisms. Then there is a normal subgroup H of G so that*

(i) *$K(x_1, \ldots, x_n)^H = K(z_1, \ldots, z_n)$ for some power products $z_1, \ldots, z_n$ in $x_1, \ldots, x_n$,*

(ii) *G/H acts on $K(z_1, \ldots, z_n)$ by monomial K-automorphisms, and*

(iii) *$\rho_{\underline{z}} : G/H \to GL_n(\mathbb{Z})$ is injective.*

Proof. Induction on the order of G.

Without loss of generality, we may assume that G acts faithfully on $K(x_1, \ldots, x_n)$. Define $H_0 = \operatorname{Ker}\{\rho_{\underline{x}} : G \to GL_n(\mathbb{Z})\}$. For any $\tau \in H_0$, $\tau(x_i) = a_i(\tau) \cdot x_i$ for some $a_i(\tau) \in K \backslash \{0\}$. In particular, H_0 is an abelian subgroup of G. Choose $\tau_1, \ldots, \tau_m \in H_0$ so that H_0 is generated by $\tau_1, \ldots, \tau_m$.

Define $\langle x_1, \ldots, x_n \rangle$ to be the multiplicative subgroup of $K(x_1, \ldots, x_n) \backslash \{0\}$ generated by $x_1, \ldots, x_n$, i.e. $\langle x_1, \ldots, x_n \rangle = \{M = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} : \lambda_i \in \mathbb{Z} \text{ for } 1 \le i \le n\}$. Define a group homomorphism

$$\begin{aligned} \Phi : \langle x_1, \ldots, x_n \rangle &\to K^\times \times K^\times \times \cdots \times K^\times \\ M &\mapsto (\tau_1(M)/M, \tau_2(M)/M, \ldots, \tau_m(M)/M) \end{aligned}$$

where $K^\times = K \backslash \{0\}$ and $M = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ with $\lambda_i \in \mathbb{Z}$. Note that the image of Φ is a finite group. It is not difficult to find that $\operatorname{Ker}(\Phi) = \langle M_1, \ldots, M_n \rangle$ where $M_1, \ldots, M_n$ are n power products in $x_1, \ldots, x_n$. Moreover, $K(x_1, \ldots, x_n)^{H_0} = K(M_1, \ldots, M_n)$.

We will show that G (equivalently, G/H_0) acts on $K(M_1, \ldots, M_n)$ by monomial K-automorphisms. For any $\sigma \in G$, any $1 \le i \le n$, $\sigma(M_i) = c_i(\sigma) \cdot N_i$ where $N_i = x_1^{v_1} \cdots x_n^{v_n}$ with $v_i \in \mathbb{Z}$ and $c_i(\sigma) \in K \backslash \{0\}$. To show that each N_i is a power product in $M_1, M_2, \ldots, M_n$, it suffices to show that $N_i \in K(x_1, \ldots, x_n)^{H_0} = K(M_1, \ldots, M_n)$. From the identity $\sigma(M_i) = c_i(\sigma) \cdot N_i$, we find that, for any $\tau \in H_0$, $c_i(\sigma) \cdot \tau(N_i) = \tau\sigma(M_i) = \sigma \cdot (\sigma^{-1}\tau\sigma)(M_i) = \sigma(M_i)$ because $\sigma^{-1}\tau\sigma \in H_0$. Thus $\tau(N_i) = N_i$ for any $\tau \in H_0$.

Now consider $\rho_{\underline{M}} : G/H_0 \to GL_n(\mathbb{Z})$. By induction hypothesis, we can find z_1, …, $z_n \in K(M_1, \ldots, M_n)$ so that (i) $z_1, \ldots, z_n$ are power products in $M_1, \ldots, M_n$, (ii) $K(z_1, \ldots, z_n) = K(M_1, \ldots, M_n)^{H/H_0}$ for some normal subgroup H/H_0 in G/H_0 (where $H \supset H_0$), (iii) G/H acts monomially on $K(z_1, \ldots, z_n)$, (iv) $\rho_{\underline{z}} : G/H \to GL_n(\mathbb{Z})$ is injective. $\square$

§3. Proof of Theorem 1.3

We will prove Theorem 1.3 in this section. Because of Lemma 2.8, it suffices to consider the situation when $\rho : G \to GL_3(\mathbb{Z})$ is injective. This condition will remain in force throughout this section.

Unless otherwise specified, we will assume, in this section, that G is a finite 2-group acting on $K(x_1, x_2, x_3)$ by monomial K-automorphisms, and K is a field satisfying the conditions (i) $\operatorname{char} K \neq 2$, and (ii) $\sqrt{a} \in K$ for any $a \in K$. For example, in the following Theorem 3.4 it is understood that K satisfies the above two conditions although there is no mention about the assumptions on K in the statement of Theorem 3.4. On the other hand, in Theorem 3.3, K satisfies the weaker conditions that $\operatorname{char} K \neq 2$ and $\sqrt{-1} \in K$, which will be stated explicitly.

Since $\rho : G \to GL_3(\mathbb{Z})$ is injective, G is isomorphic to a finite subgroup of $GL_3(\mathbb{Z})$ as an abstract group. It is known that, up to conjugation in $GL_3(\mathbb{Z})$, there are precisely 73 non-isomorphic finite subgroups in $GL_3(\mathbb{Z})$ [Ta; HK2, p.807]. We will denote by $W_i(j)$ the group W_i which appears on page j of Tahara's paper [Ta]. The 2-groups in $GL_3(\mathbb{Z})$ are the following 36 groups G,

(I) $G \simeq C_2 : W_i(173)$ where $1 \leq i \leq 5$;

(II) $G \simeq C_4 : W_i(174)$ where $1 \leq i \leq 4$; and
$G \simeq C_2 \times C_2 : W_i(174)$ where $5 \leq i \leq 15$;

(III) $G \simeq C_4 \times C_2 : W_i(187)$ where $1 \leq i \leq 2$;
$G \simeq C_2 \times C_2 \times C_2 : W_i(187)$ where $3 \leq i \leq 6$; and
$G \simeq D_4 : W_i(187)$ where $7 \leq i \leq 14$;

(IV) $G \simeq D_4 \times C_2 : W_1(194)$ and $W_2(195)$.

Convention. Suppose that $\sigma, \tau \in G$. We will adopt the convention

$$\sigma : (x_1, x_2, x_3) \mapsto (\epsilon_1 x_1, 1/x_2, 1/x_3),$$
$$\tau : (x_1, x_2, x_3) \mapsto (1/x_1, \epsilon_2/x_2, \epsilon_3/x_3),$$

to indicate the fact that σ and τ are K-automorphisms on $K(x_1, x_2, x_3)$ defined by $\sigma(x_1) = \epsilon_1 x_1$, $\sigma(x_2) = 1/x_2$, $\sigma(x_3) = 1/x_3$, $\tau(x_1) = 1/x_1$, $\tau(x_2) = \epsilon_2/x_2$, $\tau(x_3) = \epsilon_3/x_3$.

Theorem 3.1. *Let $G = W_i(173)$ for $1 \le i \le 4$ and K be a field with $\operatorname{char} K \ne 2$. Then $K(x_1, x_2, x_3)^G$ is rational over K.*

Proof. First we will "normalize" the coefficients of $\sigma(x_i)$ where $G = \langle\sigma\rangle$.

When $G = W_4(173)$, the action of $G = \langle\sigma\rangle$ is given by

$$\sigma : (x_1, x_2, x_3) \mapsto (ax_1, b/x_3, c/x_2)$$

for some $a, b, c \in K\backslash\{0\}$. Since $G \simeq C_2$, it follows that $a = \pm 1$. Define $y_2 = x_2$, $y_3 = b/x_3$. Then $K(x_2, x_3) = K(y_2, y_3)$ and $\sigma : (y_2, y_3) \mapsto (y_3, by_2/c)$. Since $G \simeq C_2$, we find that $b/c = 1$. Apply Theorem 2.1. We find that $K(x_1, x_2, x_3)^G = K(y_2, y_3, x_1)^G = K(y_2, y_3)^G(u)$ for some u with $\sigma(u) = u$. Note that $K(y_2, y_3)^G = K(y_2 + y_3, (y_2 - y_3)^2)$ is rational over K.

When $G = W_1(173)$, the action of G is given by

$$(x_1, x_2, x_3) \mapsto (ax_1, b/x_2, c/x_3)$$

for some $a, b, c \in K\backslash\{0\}$. Apply Theorem 2.1 and reduce the question to $K(x_2, x_3)^G$. Apply Theorem 1.1. Done.

When $G = W_2(173)$ and $W_3(173)$, the actions of G are given by

$$(x_1, x_2, x_3) \mapsto (a/x_1, bx_2, cx_3)$$

and

$$(x_1, x_2, x_3) \mapsto (a/x_1, bx_3, cx_2)$$

respectively where $a, b, c \in K\backslash\{0\}$.

Apply Theorem 2.1 to both cases. The questions are reduced to $K(x_1)^G$. Note that $K(x_1)^G$ is rational by Lüroth's Theorem. $\square$

Theorem 3.2. *Let K be a field with $\operatorname{char} K \ne 2$ and $G = W_5(173)$, i.e. the action of $G = \langle\sigma\rangle$ is given by*

$$\sigma : (x_1, x_2, x_3) \mapsto (a_1/x_1, a_2/x_2, a_3/x_3)$$

for some $a_1, a_2, a_3 \in K\backslash\{0\}$. Then $K(x_1, x_2, x_3)^G$ is rational over K if and only if $[K(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}) : K] \le 4$.

Proof. If $[K(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}) : K] = 8$, it is Saltman who shows that $K(x_1, x_2, x_3)^G$ is not retract rational over K [Sa] (note that K is an infinite field in this situation). In particular, $K(x_1, x_2, x_3)^G$ is not rational over K. See [Ka2] for a generalization of Saltman's Theorem.

If $[K(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}) : K] \le 4$, we may assume that $a_3 \in a_1^{\epsilon_1} a_2^{\epsilon_2} K^2$ for some $\epsilon_1, \epsilon_2 \in \{0, 1\}$. First consider the case $a_3 = a_1 a_2 b^2$ for some $b \in K\backslash\{0\}$. Define $y = x_3/(bx_1x_2)$. Then $K(x_1, x_2, x_3) = K(x_1, x_2, y)$ and $\sigma(y) = 1/y$. Define $z = (1 - y)/(1 + y)$. Then $\sigma(z) = -z$. It follows that $K(x_1, x_2, x_3)^G = K(x_1, x_2, z)^G = K(x_1, x_2)^G(u)$ for some u with $\sigma(u) = u$ by Theorem 2.1. Note that $K(x_1, x_2)^G$ is rational by Theorem 1.1. The other cases $a_3 \in a_1K^2$, $a_3 \in a_2K^2$ and $a_3 \in K^2$ can be proved similarly. $\square$

Theorem 3.3. *Let $G = W_i(174)$ for $1 \le i \le 4$ and K satisfy the conditions that* $\operatorname{char} K \neq 2$ *and* $\sqrt{-1} \in K$. *Then $K(x_1, x_2, x_3)^G$ is rational over K.*

Proof. Case 1. $G = W_1(174)$.

The action of $G = \langle \sigma \rangle$ is given by

$$\sigma : (x_1, x_2, x_3) \mapsto \big((\sqrt{-1})^i x_1, x_3, a/x_2\big)$$

for some integer i and some $a \in K \backslash \{0\}$. Apply Theorem 2.1. We find that $K(x_1, x_2, x_3)^G = K(x_2, x_3)^G(u)$ for some u with $\sigma(u) = u$. By Theorem 1.1, $K(x_2, x_3)^G$ is rational over K.

Case 2. $G = W_2(174)$.

The action of $G = \langle \sigma \rangle$ is given by

$$\sigma : (x_1, x_2, x_3) \mapsto (a/x_1, x_3, b/x_2)$$

for some $a, b \in K \backslash \{0\}$. Define u and v by

$$u = \frac{x_2 - \frac{b}{x_2}}{x_2 x_3 - \frac{b^2}{x_2 x_3}}, \quad v = \frac{x_3 - \frac{b}{x_3}}{x_2 x_3 - \frac{b^2}{x_2 x_3}}.$$

By Theorem 2.3, we get $K(x_1, x_2, x_3)^{\langle \sigma^2 \rangle} = K(x_1, u, v)$. Note that

$$\sigma : u \mapsto \frac{x_3 - \frac{b}{x_3}}{\frac{bx_3}{x_2} - \frac{bx_2}{x_3}}, \; v \mapsto \frac{-\big(x_2 - \frac{b}{x_2}\big)}{\frac{bx_3}{x_2} - \frac{bx_2}{x_3}}.$$

Note that

$$\frac{-\big(x_2 - \frac{b}{x_2}\big)}{\frac{bx_3}{x_2} - \frac{bx_2}{x_3}} = \frac{u}{bu^2 - bv^2}$$

by Theorem 2.3 again.

Define $w = u/v$. Then $K(x_1, u, v) = K(w, x_1, v)$ and

$$\sigma : (w, x_1, v) \mapsto (-1/w, a/x_1, 1/(cv))$$

where $c = b[w - (1/w)]$.

Define

$$y_1 = x_1, \quad Y_2 = \big(\sqrt{-1} - w\big)/\big(\sqrt{-1} + w\big), \quad y_2 = [x_1 - (a/x_1)]Y_2.$$

Then $K(w, x_1, v) = K(y_1, y_2, v)$ and $\sigma(y_1) = a/y_1$, $\sigma(y_2) = y_2$, $\sigma(v) = 1/(cv)$.

It is not difficult to verify that $c = b[w-(1/w)] = 2\sqrt{-1}b\big(x_1-(a/x_1)+\sqrt{-1}Y_2\big)\big(x_1-(a/x_1) - \sqrt{-1}Y_2\big)/\big[\big(x_1 - (a/x_1) + Y_2\big)\big(x_1 - (a/x_1) - Y_2\big)\big]$.

Define

$$y_3 = v(x_1 - (a/x_1) + \sqrt{-1}Y_2)/(x_1 - (a/x_1) + Y_2).$$

It follows that $\sigma(y_3)=d/y_3$ where $d=1/(2\sqrt{-1}b)$. Now we find that $K(w,x_1,v)^{\langle\sigma\rangle}$ $=K(y_1,y_2,y_3)^{\langle\sigma\rangle}=K(y_2)(y_1,y_3)^{\langle\sigma\rangle}$ is rational over $K(y_2)$ by Theorem 2.3.

Case 3. $G=W_3(174)$.
The action of $G=\langle\sigma\rangle$ is given by

$$\sigma:(x_1,x_2,x_3)\mapsto(ax_1,x_3,bx_1/x_2).$$

for some $a,b\in K\backslash\{0\}$. Define $y_1=x_2$, $y_2=x_3$, $y_3=bx_1/x_2$. Then $K(x_1,x_2,x_3)=K(y_1,y_2,y_3)$ and $\sigma:(y_1,y_2,y_3)\mapsto(y_2,y_3,\epsilon y_1y_3/y_2)$. Since $\mathrm{ord}(\sigma)=4$, it follows that $\epsilon^2=1$, i.e. $\epsilon=\pm1$.

Case 3.1. $\epsilon=1$.
The action $\sigma:(y_1,y_2,y_3)\mapsto(y_2,y_3,y_1y_3/y_2)$ is a purely monomial automorphism. Thus $K(y_1,y_2,y_3)^G$ is rational over K by Theorem 1.2.

Case 3.2. $\epsilon=-1$.
We find that $\sigma:(y_1,y_2,y_3)\mapsto(y_2,y_3,-y_1y_3/y_2)$.
Define

$$\begin{aligned} X_1&=(y_1+y_3)/2, & X_2&=[y_2-(y_1y_3/y_2)]/2,\\ X_3&=(y_1-y_3)/(2\sqrt{-1}), & X_4&=[y_2+(y_1y_3/y_2)]/(2\sqrt{-1}). \end{aligned}$$

Then $K(y_1,y_2,y_3)=K(X_1,X_2,X_3,X_4)$ with the relation $X_1^2+X_2^2+X_3^2+X_4^2=0$. Note that $\sigma:X_1\leftrightarrow X_2$, $X_3\mapsto X_4\mapsto -X_3$. Thus $\sigma^2:X_1\mapsto X_1$, $X_2\mapsto X_2$, $X_3\mapsto -X_3$, $X_4\mapsto -X_4$.

Define $Y_3=X_3/X_4$, $Y_1=X_1Y_3$, $Y_2=X_2Y_3$, $Y_4=X_3^2$. Then $K(X_1,X_2,X_3,X_4)^{\langle\sigma^2\rangle}=K(Y_1,Y_2,Y_3,Y_4)$ with the relation $Y_1^2+Y_2^2+Y_4(1+Y_3^2)=0$. Hence $Y_4\in K(Y_1,Y_2,Y_3)$. It follows that $K(y_1,y_2,y_3)^{\langle\sigma\rangle}=K(Y_1,Y_2,Y_3)^{\langle\sigma\rangle}$. Note that

$$\sigma:(Y_1,Y_2,Y_3)\mapsto(-Y_2/Y_3^2,-Y_1/Y_3^2,-1/Y_3).$$

Define $z_1=Y_1/Y_3$, $z_2=Y_2/Y_3$. Then $K(Y_1,Y_2,Y_3)=K(z_1,z_2,Y_3)$ and $\sigma:(z_1,z_2,Y_3)\mapsto(z_2,z_1,-1/Y_3)$.

Apply Theorem 2.1. We find that $K(Y_1,Y_2,Y_3)^{\langle\sigma\rangle}=K(z_1,z_2,Y_3)^{\langle\sigma\rangle}=K(Y_3)^{\langle\sigma\rangle}(u_1,u_2)$ for some u_1, u_2 with $\sigma(u_1)=u_1$, $\sigma(u_2)=u_2$.

Case 4. $G=W_4(174)$.
The action of $G=\langle\sigma\rangle$ is given by

$$\sigma:(x_1,x_2,x_3)\mapsto(c/x_1,x_3,bx_1/x_2)$$

for some $b,c\in K\backslash\{0\}$.

Define $y_1=x_2$, $y_2=x_3$, $y_3=bx_1/x_2$. Then $K(x_1,x_2,x_3)=K(y_1,y_2,y_3)$ and $\sigma:(y_1,y_2,y_3)\mapsto(y_2,y_3,a/(y_1y_2y_3))$ for some $a\in K\backslash\{0\}$.

Note that $\sigma^2 : (y_1, y_2, y_3) \mapsto (y_3, a/(y_1y_2y_3), y_1)$.

Define $t = y_1y_3$. Then $K(y_1, y_2, y_3) = K(t, y_1, y_2)$ and $\sigma^2 : (t, y_1, y_2) \mapsto (t, t/y_1, a/(ty_2))$. Define $A = t$, $B = a/t$, u and v as follows

$$u = \frac{y_1 - \frac{A}{y_1}}{y_1y_2 - \frac{AB}{y_1y_2}}, \quad v = \frac{y_2 - \frac{B}{y_2}}{y_1y_2 - \frac{AB}{y_1y_2}}.$$

By Theorem 2.3, we find that $K(t, y_1, y_2)^{\langle\sigma^2\rangle} = K(t, u, v)$. Note that

$$\begin{aligned} \sigma : t &\mapsto a/t, \\ u &\mapsto \frac{y_2 - \frac{B}{y_2}}{\frac{Ay_2}{y_1} - \frac{By_1}{y_2}}, \ v \mapsto \frac{-\left(y_1 - \frac{A}{y_1}\right)}{\frac{Ay_2}{y_1} - \frac{By_1}{y_2}}. \end{aligned}$$

Note that

$$\frac{-\left(y_1 - \frac{A}{y_1}\right)}{\frac{Ay_2}{y_1} - \frac{By_1}{y_2}} = \frac{u}{Bu^2 - Av^2}$$

by Theorem 2.3.

Define $w = u/v$. Then $K(t, u, v) = K(t, w, v)$ and $\sigma : (t, w, v) \mapsto (a/t, -1/w, C/v)$ where $C = w/(Bw^2 - A)$.

Define $Z_1 = (\sqrt{-1} - w)/(\sqrt{-1} + w)$, $z_2 = w/t$. Then $\sigma(Z_1) = -Z_1$, $\sigma(z_2) = -1/(az_2)$.

Define $z_1 = Z_1[z_2 + 1/(az_2)]$. Then $\sigma : (z_1, z_2, v) \mapsto (z_1, -1/(az_2), C/v)$ where $C = w/(Bw^2 - A) = 1/[(aw/t) - (t/w)] = 1/[az_2 - (1/z_2)]$.

Define $z_3 = v[az_2 - (1/z_2)]$. Then $K(t, u, v) = K(t, w, v) = K(z_1, z_2, z_3)$ and $\sigma : (z_1, z_2, z_3) \mapsto (z_1, -1/(az_2), D/z_3)$ where $D = az_2 - (1/z_2) = a[z_2 + \sigma(z_2)]$. By Theorem 2.4 (instead of Theorem 2.3 !), $K(z_1, z_2, z_3)^{\langle\sigma\rangle}$ is rational over $K(z_1)$. ☐

Theorem 3.4. *Let $G = W_i(174)$ for $5 \le i \le 15$. Then $K(x_1, x_2, x_3)^G$ is rational over K.*

Proof. Case 1. $G = W_5(174)$.

The action of $G = \langle\sigma, \tau\rangle$ is given by

$$\begin{aligned} \sigma &: (x_1, x_2, x_3) \mapsto (\epsilon x_1, b_2/x_2, b_3/x_3), \\ \tau &: (x_1, x_2, x_3) \mapsto (a_1/x_1, a_2/x_2, a_3/x_3), \end{aligned}$$

for some $a_1, a_2, a_3, b_2, b_3 \in K\backslash\{0\}$ and $\epsilon = \pm 1$.

Since $\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3} \in K$, we may define $y_i = x_i/\sqrt{a_i}$ for $1 \le i \le 3$. It follows that $\sigma : (y_1, y_2, y_3) \mapsto (\epsilon y_1, c_2/y_2, c_3/y_3)$ and $\tau : (y_1, y_2, y_3) \mapsto (1/y_1, 1/y_2, 1/y_3)$ for some $c_2, c_3 \in K\backslash\{0\}$. Since $\sigma\tau = \tau\sigma$, it follows that $c_2^2 = c_3^2 = 1$.

If $c_2c_3 = 1$, we define $y_4 = y_2/y_3$. It follows that $\sigma(y_4) = \tau(y_4) = 1/y_4$. Define $y_5 = (1 - y_4)/(1 + y_4)$. Then $\sigma(y_5) = \tau(y_5) = -y_5$. Thus $K(y_1, y_2, y_3)^G = K(y_1, y_2, y_5)^G = K(y_1, y_2)^G(y_0)$ for some y_0 with $\sigma(y_0) = \tau(y_0) = y_0$ by Theorem 2.1. Now $K(y_1, y_2)^G$ is rational by Theorem 1.1.

If $c_2c_3 = -1$, we may assume that $c_2 = -1$ and $c_3 = 1$. In this situation, define $y_4 = y_3$. The arguments in the preceding paragraph remain valid. Done.

It is can be shown that $K(x_1, x_2, x_3)^G$ is rational when $G = W_5(174)$ and K is any field with char $K \neq 2$, i.e. the assumption that $\sqrt{a} \in K$ for any $a \in K$ can be waived. But we omit the proof here.

Case 2. $G = W_6(174)$.

The action of $G = \langle \sigma, \tau \rangle$ is given by

$$\begin{aligned} \sigma &: (x_1, x_2, x_3) \mapsto (\epsilon_1 x_1, a_2/x_2, a_3/x_3), \\ \tau &: (x_1, x_2, x_3) \mapsto (a_1/x_1, a_4/x_2, \epsilon_3 x_3), \end{aligned}$$

for some $a_1, a_2, a_3, a_4 \in K\backslash\{0\}$ and $\epsilon_1, \epsilon_3 \in \{1, -1\}$.

Define $y_i = x_i/\sqrt{a_i}$ for $1 \leq i \leq 3$. Using the relation $\sigma\tau = \tau\sigma$, we get

$$\begin{aligned} \sigma &: (y_1, y_2, y_3) \mapsto (\epsilon_1 y_1, 1/y_2, 1/y_3), \\ \tau &: (y_1, y_2, y_3) \mapsto (1/y_1, \epsilon_2/y_2, \epsilon_3 y_3) \end{aligned}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{1, -1\}$.

If at least one of $\epsilon_1, \epsilon_2, \epsilon_3$ is 1, say $\epsilon_3 = 1$, define $y_4 = (1 - y_3)/(1 + y_3)$. Then we get $\sigma(y_3) = -y_3$ and $\sigma(y_3) = y_3$. Then we may apply Theorem 2.1 and get $K(y_1, y_2, y_3)^G = K(y_1, y_2, y_4)^G = K(y_1, y_2)^G(y_0)$ with $\sigma(y_0) = \tau(y_0) = y_0$.

It remains to consider the situation $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$.

Define z_1, u and v by

$$z_1 = y_1(y_2 - (1/y_2)), \quad u = \frac{y_2 - \frac{1}{y_2}}{y_2y_3 - \frac{1}{y_2y_3}}, \quad v = \frac{y_3 - \frac{1}{y_3}}{y_2y_3 - \frac{1}{y_2y_3}}.$$

Then $K(y_1, y_2, y_3)^{\langle\sigma\rangle} = K(z_1, u, v)$ by Theorem 2.3. Define $w = u/v$. Use the same technique as in the proof of Case 2 of Theorem 3.3. We find that

$$\tau : (z_1, u, w) \mapsto (A/z_1, B/u, -w)$$

where $A = [y_2 - (1/y_2)]^2 = [(u^2 + w^2 - u^2w^2)^2/(u^2w^2)] - 4$, $B = -w^2/(w^2 - 1)$. Note that $\tau(A) = A$.

Define $z_2 = u(w - 1)/w$. Then $K(z_1, u, v) = K(z_1, z_2, w)$ and $\tau(z_2) = -1/z_2$. It is not difficult to verify that

$$\begin{aligned} A &= z_2 - (1/z_2) + z_2(1 + w)^2 - [(1 - w)^2/z_2] \\ &= 2w(z_2 + (1/z_2)) + (2 + w^2)(z_2 - (1/z_2)). \end{aligned}$$

Define $t = (\sqrt{-1} - z_2)/(\sqrt{-1} + z_2)$. Then $K(z_1, u, w) = K(t, w, z_1)$ and

$$\tau : (t, w, z_1) \mapsto (-t, -w, A/z_1) \tag{3.1}$$

where $A = 2\sqrt{-1}[w^2(1+t^2) - 4wt + 2(1+t^2)]/(1-t^2)$.

Define $x = tw$, $y = z_1(1-t)/e$ where $e \in K$ satisfies $e^2 = 2\sqrt{-1}$. We will use Theorem 2.5 to show that $K(t,x,y)^{\langle\tau\rangle}$ is rational over $K(t^2)$; in particular, $K(t,x,y)^{\langle\tau\rangle}$ is rational over K.

Note that $\tau(t) = -t$, $\tau(x) = x$, $\tau(y) = B/y$ where

$$\begin{aligned} B &= w^2(1+t^2) - 4wt + 2(1+t^2) \\ &= x^2[1+(1/t^2)] - 4x + 2(1+t^2) \\ &= [1+(1/t^2)]\{[x - (2t^2/(1+t^2))]^2 + (2t^2+2t^6)/(1+t^2)^2\}. \end{aligned}$$

In applying Theorem 2.5, define $a = t^2$, $F = K(t^2)$, $b = (1+t^2)/t^2 = (1+a)/a$ and $c = -(2a+2a^3)/(1+a)^2$. It is known that $K(t,x,y)^{\langle\tau\rangle} = F(t)(x,y)^{\langle\tau\rangle}$ is rational over F if and only if the Hilbert symbol $(a,b)_2$ is trivial in the field $F(\sqrt{ac})$.

We claim that $F(\sqrt{ac})$ is a rational function field of one variable over K. Note that $F(\sqrt{ac}) = F(\sqrt{1+a^2})$ because $\sqrt{-2} \in K \subset F$. Write $p = \sqrt{1+a^2}$, $s = p - a$. Then $F(\sqrt{1+a^2}) = K(a,p)$. From the relation $(p-a)(p+a) = 1$, we find that $p + a = 1/(p-a) = 1/s \in K(s)$. Thus $a = (1-s^2)/(2s) \in K(s)$. Hence $F(\sqrt{1+a^2}) = K(a,p) = K(s)$.

Now we find that $(a,b)_2 = (a,(1+a)/a)_2 = (a, a(1+a))_2 = (a, -(1+a))_2 = (a, 1+a)_2$ because $-1 \in K^2$. To show that $(a, 1+a)_2 = 0$ is equivalent to find a solution in $F(\sqrt{ac}) = K(s)$ of the equation

$$aX^2 + (1+a)Y^2 = 1.$$

Substitute $a = (1-s^2)/(2s)$ into the above equation. We get

$$(1-s^2)X^2 + (1+2s-s^2)Y^2 = 2s.$$

Note that $X = \sqrt{-1}$, $Y = 1$ is a solution we need. Hence the result.

It can be shown that the field $F(t)(x,y)^{\langle\tau\rangle}$ is the function field of certain conic bundle of $\mathbb{P}^1$ defined over F. Thus we may as well use Iskovskikh's theory of conic bundles [Is] to show that $F(t)(x,y)^{\langle\tau\rangle}$ is rational over F. See also [Ka3].

Case 3. $G = W_7(174)$.

The action of $G = \langle\sigma,\tau\rangle$ is given by

$$\begin{aligned} \sigma &: (x_1,x_2,x_3) \mapsto (\epsilon x_1, a_2/x_2, a_3/x_3) \\ \tau &: (x_1,x_2,x_3) \mapsto (\epsilon_1 x_1, \epsilon_2 x_2, b/x_3) \end{aligned}$$

where $a_2, a_3, b \in K\backslash\{0\}$ and $\epsilon, \epsilon_1, \epsilon_2 \in \{1,-1\}$.

Apply Theorem 2.1. The question is reduced to the rationality of $K(x_2,x_3)^G$. Done.

Case 4. $G = W_8(174)$.

The action of $G = \langle \sigma, \tau \rangle$ is given by

$$\sigma : (x_1, x_2, x_3) \mapsto (\epsilon x_1, a_2/x_2, a_3/x_3),$$
$$\tau : (x_1, x_2, x_3) \mapsto (b_1/x_1, b_2/x_3, b_3/x_2)$$

where $\epsilon_1, a_2, a_3, b_1, b_2, b_3 \in K\backslash\{0\}$.

Define $y_1 = x_1/\sqrt{b_1}$, $y_2 = x_2/\sqrt{a_2}$, $y_3 = x_3/\sqrt{a_3}$. We find that

$$\sigma : (y_1, y_2, y_3) \mapsto (\epsilon_1 y_1, 1/y_2, 1/y_3),$$
$$\tau : (y_1, y_2, y_3) \mapsto (1/y_1, \epsilon_2/y_2, \epsilon_3/y_3)$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{1, -1\}$.

The situation is the same as in Case 1. Hence we omit the proof.

Case 5. $G = W_9(174)$.

The action of $G = \langle \sigma, \tau \rangle$ is given by

$$\sigma : (x_1, x_2, x_3) \mapsto (a_1 x_1, a_2/x_2, a_3/x_3)$$
$$\tau : (x_1, x_2, x_3) \mapsto (b_1 x_1, b_2 x_3, b_3 x_2)$$

where $a_i, b_j \in K\backslash\{0\}$.

Apply Theorem 2.1. The question is reduced to the rationality of $K(x_2, x_3)^G$.

Case 6. $G = W_{10}(174)$.

The action of $G = \langle \sigma, \tau \rangle$ is given by

$$\sigma : (x_1, x_2, x_3) \mapsto (a_1/x_1, a_2 x_2, a_3 x_3),$$
$$\tau : (x_1, x_2, x_3) \mapsto (b_1 x_1, b_2 x_3, b_3 x_2)$$

where $a_i, b_j \in K\backslash\{0\}$.

Define $y = x_1/\sqrt{a_1}$. Then $\sigma(y) = 1/y$ and $\tau(y) = \pm y$.

If $\tau(y) = -y$, we will apply Theorem 2.1 and reduce the question to the rationality of $K(y)^G$.

If $\tau(y) = y$, define $z = (1-y)/(1+y)$. Then $\sigma(z) = -z$ and $\tau(z) = z$. Apply Theorem 2.1 and reduced the question to the rationality of $K(x_2, x_3)^G$.

Case 7. $G = W_{11}(174)$.

After the "normalization" of the coefficients, the action of $G = \langle \sigma, \tau \rangle$ is given by

$$\sigma : (x_1, x_2, x_3) \mapsto (1/x_1, x_3, x_2),$$
$$\tau : (x_1, x_2, x_3) \mapsto (\epsilon x_1, b/x_2, b/x_3)$$

where $b \in K\backslash\{0\}$ and $\epsilon = \pm 1$.

Define $y_i = x_i/\sqrt{b}$ for $i = 2, 3$. Then $\sigma : y_2 \leftrightarrow y_3$ and $\tau : y_2 \mapsto 1/y_2$, $y_3 \mapsto 1/y_3$.

If $\tau(x_1) = -x_1$, we may apply Theorem 2.1 and consider the rationality of $K(x_1)^G$.

If $\tau(x_1) = x_1$, define $y_1 = (1 - x_1)/(1 + x_1)$. We get $\sigma(y_1) = -y_1$ and $\tau(y_1) = y_1$. Apply Theorem 2.1 and consider the rationality of $K(y_2, y_3)^G$. Done.

Case 8. $G = W_{12}(174)$.

The action of $G = \langle \sigma, \tau \rangle$ is given by

$$\begin{aligned}
\sigma &: (x_1, x_2, x_3) \mapsto (1/x_1, x_3, x_2), \\
\tau &: (x_1, x_2, x_3) \mapsto (b_1 x_2/(x_1 x_3), b_2/x_3, b_3/x_2)
\end{aligned}$$

where $b_1, b_2, b_3 \in K \backslash \{0\}$.

Define $y_1 = x_1$, $y_2 = x_2/\sqrt{b_2}$, $y_3 = x_3/\sqrt{b_2}$. Use the fact that $\sigma\tau = \tau\sigma$. We find that

$$\begin{aligned}
\sigma &: (y_1, y_2, y_3) \mapsto (1/y_1, y_3, y_2), \\
\tau &: (y_1, y_2, y_3) \mapsto (\epsilon y_2/(y_1 y_3), 1/y_3, 1/y_2)
\end{aligned}$$

where $\epsilon = \pm 1$.

If $\epsilon = 1$, apply Theorem 1.2. Thus $K(y_1, y_2, y_3)^G$ is rational.

It remains to consider the case that $\epsilon = -1$.

Define $t = y_2/y_3$. Then $K(y_1, y_2, y_3) = K(t, y_1, y_2)$ and

$$\begin{aligned}
\sigma &: (t, y_1, y_2) \mapsto (1/t, 1/y_1, y_2/t), \\
\tau &: (t, y_1, y_2) \mapsto (t, -t/y_1, t/y_2).
\end{aligned}$$

Define

$$u = \frac{y_1 + \frac{t}{y_1}}{y_1 y_2 + \frac{t^2}{y_1 y_2}}, \quad v = \frac{y_2 - \frac{t}{y_2}}{y_1 y_2 + \frac{t^2}{y_1 y_2}}, \quad w = u/v.$$

By Theorem 2.3 we find that $K(t, y_1, y_2)^{\langle \tau \rangle} = K(t, u, w)$ and

$$\sigma : t \mapsto 1/t, \ w \mapsto w, \ u \mapsto A/u$$

where $A = w^2/(1 + w^2)$ (the computation is similar to Case 4).

Define $s = [(1 - t)/(1 + t)][u - (A/u)]$. Then $K(t, u, w) = K(s, u, w)$ and $\sigma(s) = s$. Thus we may reduce the question to the rationality of $K(w)(u)^{\langle \sigma \rangle}$, which is easy by Lüroth's Theorem. Hence the result.

Case 9. $G = W_{13}(174)$.

The action of $G = \langle \sigma, \tau \rangle$ is given by

$$\begin{aligned}
\sigma &: (x_1, x_2, x_3) \mapsto (1/x_1, x_3, x_2), \\
\tau &: (x_1, x_2, x_3) \mapsto (a_1 x_1 x_3/x_2, a_2 x_3, a_3 x_2)
\end{aligned}$$

where $a_1, a_2, a_3 \in K \backslash \{0\}$.

Define $x_0 = x_2/x_3$. Then $K(x_1, x_2, x_3) = K(x_0, x_1, x_2)$ and

$$\begin{aligned}
\sigma &: (x_0, x_1, x_2) \mapsto (1/x_0, 1/x_1, x_2/x_0), \\
\tau &: (x_0, x_1, x_2) \mapsto (a_2/(a_3x_0), a_1x_1/x_0, a_2x_2/x_0)
\end{aligned}$$

Apply Theorem 2.2. It suffices to show that $K(x_0, x_1)^G$ is rational. But this follows from Theorem 1.1.

Case 10. $G = W_{14}(174)$.
The action of $G = \langle \sigma, \tau \rangle$ is given by

$$\begin{aligned}
\sigma &: (x_1, x_2, x_3) \mapsto (1/x_1, x_3, x_2), \\
\tau &: (x_1, x_2, x_3) \mapsto (a_1/x_1, a_2x_1/x_3, a_3/(x_1x_2))
\end{aligned}$$

where $a_1, a_2, a_3 \in K\backslash\{0\}$.

Since $\tau^2 = 1$, we find that $a_3 = a_1a_2$. From $\sigma\tau(x_2) = \tau\sigma(x_2)$, we find that $a_2 = a_3$. Thus $a_1 = 1$. In summary, we have that

$$\begin{aligned}
\sigma &: (x_1, x_2, x_3) \mapsto (1/x_1, x_3, x_2), \\
\tau &: (x_1, x_2, x_3) \mapsto (1/x_1, ax_1/x_3, a/(x_1x_2))
\end{aligned}$$

where $a \in K\backslash\{0\}$.

Define $y_i = x_i/\sqrt{a}$ for $i = 2, 3$. Then $\sigma : y_2 \leftrightarrow y_3$, $\tau : y_2 \mapsto x_1/y_3$, $y_3 \mapsto 1/(x_1y_2)$. Thus G acts on $K(x_1, y_2, y_3)$ by purely monomial K-automorphisms. Hence we may apply Theorem 1.2.

Case 11. $G = W_{15}(174)$.
The action of $G = \langle \sigma, \tau \rangle$ is given by

$$\begin{aligned}
\sigma &: (x_1, x_2, x_3) \mapsto (1/x_1, x_3, x_2), \\
\tau &: (x_1, x_2, x_3) \mapsto (a_1x_1, a_2x_3/x_1, a_3x_1x_2).
\end{aligned}$$

Define $x_0 = x_2/x_3$. The proof is almost the same as that of Case 9. $G = W_{13}(174)$. $\square$

Theorem 3.5. *Let $G = W_i(187)$ for $1 \le i \le 2$. Then $K(x_1, x_2, x_3)^G$ is rational over K.*

Proof. Case 1. $G = W_1(187)$.
The action of $G = \langle \sigma, \tau \rangle$ is given by

$$\begin{aligned}
\sigma &: (x_1, x_2, x_3) \mapsto \left((\sqrt{-1})^i x_1, x_3, b/x_2\right), \\
\tau &: (x_1, x_2, x_3) \mapsto (c_1/x_1, c_2/x_2, c_3/x_3)
\end{aligned}$$

where $b, c_1, c_2, c_3 \in K\backslash\{0\}$ and $i \in \{0, 1, 2, 3\}$.

Since $\sigma\tau = \tau\sigma$, we find that $i = 0$ or 2, $c_2 = c_3$, $(b/c_2)^2 = 1$. In summary, we have

$$\begin{aligned}\sigma &: (x_1, x_2, x_3) \mapsto (\epsilon_1 x_1, x_3, a_2\epsilon_2/x_2),\\ \tau &: (x_1, x_2, x_3) \mapsto (a_1/x_1, a_2/x_2, a_2/x_3)\end{aligned}$$

where $a_1, a_2 \in K\backslash\{0\}$ and $\epsilon_1, \epsilon_2 \in \{1, -1\}$.

Define $y_1 = x_1/\sqrt{a_1}$, $y_2 = x_2/\sqrt{a_2}$, $y_3 = x_3/\sqrt{a_2}$. We get that

$$\begin{aligned}\sigma &: (y_1, y_2, y_3) \mapsto (\epsilon_1 y_1, y_3, \epsilon_2/y_2),\\ \tau &: (y_1, y_2, y_3) \mapsto (1/y_1, 1/y_2, 1/y_3)\end{aligned}$$

where $\epsilon_1, \epsilon_2 \in \{1, -1\}$.

If $\epsilon_1 = 1$, define $z_1 = (1 - y_1)/(1 + y_1)$. Then $\sigma(z_1) = z_1$, $\tau(z_1) = -z_1$. Apply Theorem 2.1 and reduce the question to $K(y_2, y_3)^G$.

If $\epsilon_1 = -1$ and $\epsilon_2 = 1$, define $z_2 = (1 - y_2)/(1 + y_2)$, $z_3 = (1 - y_3)/(1 + y_3)$. Then $\sigma(z_2) = z_3$, $\sigma(z_3) = -z_2$, $\tau(z_2) = -z_2$, $\tau(z_3) = -z_3$. Apply Theorem 2.1 and reduce the question to $K(y_1)^G$.

It remains to consider the situation when $\epsilon_1 = \epsilon_2 = -1$.

Note that $\sigma^2(y_1) = y_1$, $\sigma^2(y_2) = -1/y_2$, $\sigma^2(y_3) = -1/y_3$. Define

$$u = \frac{y_2 + \frac{1}{y_2}}{y_2 y_3 - \frac{1}{y_2 y_3}}, \quad v = \frac{y_3 + \frac{1}{y_3}}{y_2 y_3 - \frac{1}{y_2 y_3}}, \quad w = u/v.$$

By Theorem 2.3 we find that $K(y_1, y_2, y_3)^{\langle\sigma^2\rangle} = K(y_1, u, v) = K(y_1, w, v)$ and

$$\begin{aligned}\sigma &: (y_1, w, v) \mapsto (-y_1, -1/w, A/v)\\ \tau &: (y_1, w, v) \mapsto (1/y_1, w, -v)\end{aligned}$$

where $A = w/(w^2 - 1)$.

Define $z_1 = y_1 + (1/y_1)$, $z_2 = v[y_1 - (1/y_1)]$. Then $K(y_1, w, v)^{\langle\tau\rangle} = K(z_1, z_2, w)$ and

$$\sigma : z_1 \mapsto -z_1, \ w \mapsto -1/w, \ z_2 \mapsto B/z_2$$

where $B = (z_1^2 - 4)/[(1/w) - w]$.

Define $t = z_1[w + (1/w)]$, $z_3 = z_2[(1/w) - w]/(z_1 - 2)$. Then $K(z_1, z_2, w) = K(t, w, z_3)$ and

$$\sigma : t \mapsto t, \ w \mapsto -1/w, \ z_3 \mapsto [w - (1/w)]/z_3.$$

Thus $K(t, w, z_3)^{\langle\sigma\rangle}$ is rational over $K(t)$ by Theorem 2.4 (instead of Theorem 2.3).

Case 2. $G = W_2(187)$.

The action of $G = \langle\sigma, \tau\rangle$ is given by

$$\begin{aligned}\sigma &: (x_1, x_2, x_3) \mapsto (a_1 x_1, x_3, a_2 x_1/x_2),\\ \tau &: (x_1, x_2, x_3) \mapsto (b_1/x_1, b_2/x_2, b_3/x_3)\end{aligned}$$

where $a_i, b_j \in K\backslash\{0\}$.

Define $y_1 = x_2$, $y_2 = x_3$, $y_3 = a_2 x_1/x_2$. Using the relation $\sigma^4 = 1$, $\sigma\tau = \tau\sigma$, we get

$$\begin{aligned} \sigma &: (y_1, y_2, y_3) \mapsto (y_2, y_3, \epsilon y_1 y_3/y_2), \\ \tau &: (y_1, y_2, y_3) \mapsto (a/y_1, a/y_2, a/y_3) \end{aligned}$$

where $\epsilon \in \{1, -1\}$ and $a \in K\backslash\{0\}$.

Define $z_i = y_i/\sqrt{a}$ for $1 \le i \le 3$. Then we have

$$\begin{aligned} \sigma &: (z_1, z_2, z_3) \mapsto (z_2, z_3, \epsilon z_1 z_3/z_2), \\ \tau &: (z_1, z_2, z_3) \mapsto (1/z_1, 1/z_2, 1/z_3). \end{aligned}$$

where $\epsilon \in \{1, -1\}$.

If $\epsilon = 1$, the action of G is a purely monomial action. Apply Theorem 1.2.

It remains to consider the situation when $\epsilon = -1$.

Define $t = z_1 z_3$. Then $K(z_1, z_2, z_3) = K(t, z_1, z_2)$ and

$$\begin{aligned} \sigma &: (t, z_1, z_2) \mapsto (-t, z_2, t/z_1), \\ \tau &: (t, z_1, z_2) \mapsto (1/t, 1/z_1, 1/z_2). \end{aligned}$$

Note that $\sigma^2(t) = t$, $\sigma^2(z_1) = t/z_1$, $\sigma^2(z_2) = -t/z_2$. Define

$$u = \frac{z_1 - \frac{t}{z_1}}{z_1 z_2 + \frac{t^2}{z_1 z_2}}, \quad v = \frac{z_2 + \frac{t}{z_2}}{z_1 z_2 + \frac{t^2}{z_1 z_2}}, \quad w = u/v.$$

By Theorem 2.3 we find that $K(t, z_1, z_2)^{\langle\sigma^2\rangle} = K(t, u, v) = K(t, w, v)$ and

$$\begin{aligned} \sigma &: (t, w, v) \mapsto (-t, -1/w, A/v) \\ \tau &: (t, w, v) \mapsto (1/t, -w, tv) \end{aligned}$$

where $A = -w/[t(w^2 + 1)]$.

Define $u_1 = t + (1/t)$, $u_2 = w[t - (1/t)]$, $u_3 = v(1 + t)$. Then $K(t, w, v)^{\langle\tau\rangle} = K(u_1, u_2, u_3)$ and

$$\sigma : u_1 \mapsto -u_1, \ u_2 \mapsto (u_1^2 - 4)/u_2, \ u_3 \mapsto B/u_3$$

where $B = A(1 - t^2) = w(t^2 - 1)/[t(w^2 + 1)] = u_2(u_1^2 - 4)/[u_2^2 + (u_1^2 - 4)]$.

Define $v_1 = u_1$, $v_2 = u_2/(u_1 - 2)$, $v_3 = u_3$. Then

$$\sigma : v_1 \mapsto -v_1, \ v_2 \mapsto -1/v_2, \ v_3 \mapsto B/v_3$$

where $B = (v_1 + 2)v_2/[v_2^2 + ((v_1 + 2)/(v_1 - 2))]$.

Define $v_4 = (\sqrt{-1} - v_2)/(\sqrt{-1} + v_2)$. Then $\sigma(v_4) = -v_4$. It is not difficult to verify that

$$B = (v_1^2 - 4)(v_4^2 - 1)/[4\sqrt{-1}(v_4^2 + v_1 v_4 + 1)].$$

Define $s = v_1^2$, $\alpha = v_1$, $x = v_1 v_4$, $y = e \cdot v_3(v_4^2 + v_1 v_4 + 1)/[(v_1 - 2)(v_4 - 1)]$ where $e \in K$ satisfying $e^2 = 4\sqrt{-1}$. Note that $K(u_1, u_2, u_3) = K(\alpha, x, y)$ and we find that

$$\sigma : \alpha \mapsto -\alpha,\ x \mapsto x,\ y \mapsto f(x)/y \tag{3.2}$$

where $f(x) = \frac{1}{s}\{[x + (s/2)]^2 - [(s^2 - 4s)/4]\}$.

We will apply Theorem 2.5 (note that Formula (3.2) looks very similar to Formula (3.1) in the proof of Case 2 in Theorem 3.4). In the notation of Theorem 2.5, $F = K(s)$ and $E = F(\alpha)$. It remains to show that the Hilbert symbol $(s, 1/s)_2$ is trivial in $K(s, \sqrt{s-4})$. Since $-1 \in K^2$, we find that $(s, 1/s)_2 = (s, s)_2 = (s, -s)_2 = 0$. Hence $K(\alpha, x, y)^{\langle\sigma\rangle} = K(s)(\alpha, x, y)^{\langle\sigma\rangle}$ is rational over $K(s)$. □

Theorem 3.6. *Let $G = W_i(187)$ for $3 \le i \le 6$. Then $K(x_1, x_2, x_3)^G$ is rational over K.*

Proof. Case 1. $G = W_3(187)$.

The action of $G = \langle\sigma, \tau, \lambda\rangle$ is given by

$$\begin{aligned}
\sigma &: (x_1, x_2, x_3) \mapsto (\epsilon_1 x_1, 1/x_2, 1/x_3),\\
\tau &: (x_1, x_2, x_3) \mapsto (1/x_1, \epsilon_2/x_2, \epsilon_3/x_3),\\
\lambda &: (x_1, x_2, x_3) \mapsto (a_1/x_1, a_2/x_2, a_3/x_3)
\end{aligned}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{1, -1\}$, $a_1, a_2, a_3 \in K\backslash\{0\}$ (for the actions of σ and τ, compare with the actions of σ and τ for $G = W_6(174)$ in the proof of Theorem 3.4).

Since $\sigma\lambda = \lambda\sigma$ and $\tau\lambda = \lambda\tau$, we find that $a_1^2 = a_2^2 = a_3^2 = 1$.

We will consider only the case $\epsilon_1 = -1$, because the case $\epsilon_1 = 1$ is easier and can be proved similarly.

Define $y_1 = x_1$ and

$$y_i = \frac{1 - x_i}{1 + x_i}$$

for $i = 2, 3$. Thus we get

$$\begin{aligned}
\sigma &: (y_1, y_2, y_3) \mapsto (-y_1, -y_2, -y_3),\\
\tau &: (y_1, y_2, y_3) \mapsto (y_1^{-1}, -y_2^{\epsilon_2}, -y_3^{\epsilon_3}),\\
\lambda &: (y_1, y_2, y_3) \mapsto (a_1 y_1^{-1}, -y_2^{a_2}, -y_3^{a_3})
\end{aligned}$$

where $\epsilon_2, \epsilon_3, a_1, a_2, a_3 \in \{1, -1\}$.

Define $z_1 = y_1/y_2$, $z_2 = y_2/y_3$, $z_3 = y_3^2$. Then $K(y_1, y_2, y_3)^{\langle\sigma\rangle} = K(z_1, z_2, z_3)$. It is not difficult to see that anyone of $\tau(z_i)$, $\lambda(z_j)$ where $1 \le i, j \le 3$ is of the form

$$\epsilon z_1^{b_1} z_2^{b_2} z_3^{b_3}$$

where $\epsilon \in \{1, -1\}$ and $b_1, b_2, b_3 \in \mathbb{Z}$. Hence $\langle\tau, \lambda\rangle$ acts on $K(z_1, z_2, z_3)$ by monomial K-automorphisms. Since $\langle\tau, \lambda\rangle \simeq C_2 \times C_2$, we find that $K(z_1, z_2, z_3)^{\langle\tau,\lambda\rangle}$ is rational over K by Theorem 3.4. Done.

Case 2. $G = W_4(187)$.

The action of $G = \langle \sigma, \tau, \lambda \rangle$ is given by

$$\begin{aligned}
\sigma &: (x_1, x_2, x_3) \mapsto (a_1 x_1, a_2/x_2, a_3/x_3), \\
\tau &: (x_1, x_2, x_3) \mapsto (b_1/x_1, b_2/x_3, b_3/x_2), \\
\lambda &: (x_1, x_2, x_3) \mapsto (c_1/x_1, c_2/x_2, c_3/x_3)
\end{aligned}$$

where $a_i, b_j, c_k \in K \backslash \{0\}$.

Define $y_i = x_i/\sqrt{c_i}$ for $1 \le i \le 3$. It is easy to see that

$$\begin{aligned}
\sigma &: (y_1, y_2, y_3) \mapsto (\epsilon_1 y_1, \epsilon_2/y_2, \epsilon_3/y_3), \\
\tau &: (y_1, y_2, y_3) \mapsto (e_1/y_1, e_2/y_3, e_3/y_2), \\
\lambda &: (y_1, y_2, y_3) \mapsto (1/y_1, 1/y_2, 1/y_3)
\end{aligned}$$

where $\epsilon_1, \epsilon_2, \epsilon_3, e_1, e_2, e_3 \in \{1, -1\}$ and $e_2 = e_3$.

Define $z_i = (1 - y_i)/(1 + y_i)$ for $1 \le i \le 3$. Consider $K(z_1, z_2, z_3)^{\langle \lambda \rangle}$ and then proceed as in the above Case 1. The details are left to the reader.

Case 3. $G = W_5(187)$.

The action of $G = \langle \sigma, \tau, \lambda \rangle$ is given by

$$\begin{aligned}
\sigma &: (x_1, x_2, x_3) \mapsto (1/x_1, x_3, x_2), \\
\tau &: (x_1, x_2, x_3) \mapsto (\epsilon x_2/(x_1 x_3), 1/x_3, 1/x_2), \\
\lambda &: (x_1, x_2, x_3) \mapsto (a_1/x_1, a_2/x_2, a_3/x_3)
\end{aligned}$$

where $\epsilon \in \{1, -1\}$, $a_1, a_2, a_3 \in K \backslash \{0\}$ (for the actions of σ and τ, compare with those for $G = W_{12}(174)$ in the proof of Theorem 3.4).

Since $G \simeq C_2 \times C_2 \times C_2$, it is easy to see that $a_2 = a_3$ and $a_1, a_2 \in \{1, -1\}$.

Define $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_2/x_3$. We find that

$$\begin{aligned}
\sigma &: (y_1, y_2, y_3) \mapsto (1/y_1, y_2/y_3, 1/y_3), \\
\tau &: (y_1, y_2, y_3) \mapsto (\epsilon y_3/y_1, y_3/y_2, y_3), \\
\lambda &: (y_1, y_2, y_3) \mapsto (\epsilon_1/y_1, \epsilon_2/y_2, 1/y_3)
\end{aligned}$$

where $\epsilon, \epsilon_1, \epsilon_2 \in \{1, -1\}$.

Define

$$u = \frac{y_1 - \frac{\epsilon y_3}{y_1}}{y_1 y_2 - \frac{\epsilon y_3^2}{y_1 y_2}}, \quad v = \frac{y_2 - \frac{y_3}{y_2}}{y_1 y_2 - \frac{\epsilon y_3^2}{y_1 y_2}}, \quad w = u/v.$$

By Theorem 2.3, we get $K(y_1, y_2, y_3)^{\langle \tau \rangle} = K(u, v, y_3)$ and

$$\begin{aligned}
\sigma &: (u, w, y_3) \mapsto (A/u, -\epsilon w, 1/y_3), \\
\lambda &: (u, w, y_3) \mapsto (\epsilon_2 y_3 u, \epsilon \epsilon_1 \epsilon_2 w, 1/y_3)
\end{aligned}$$

where $A = w^2/(w^2 - \epsilon)$.

Define $z_1 = w$, $z_2 = (1 - y_3)/(1 + y_3)$, $z_3 = 2uy_3/(1 + y_3)$. Then $K(u, w, y_3) = K(z_1, z_2, z_3)$ and

$$\begin{aligned} \sigma &: (z_1, z_2, z_3) \mapsto (-\epsilon z_1, -z_2, B/z_3), \\ \lambda &: (z_1, z_2, z_3) \mapsto (\epsilon\epsilon_1\epsilon_2 z_1, -z_2, \epsilon_2 z_3) \end{aligned}$$

where $B = A(1 - z_2^2) = z_1^2(1 - z_2^2)/(z_1^2 - \epsilon)$.

Case 3.1. $\epsilon_2 = -1$ and $\epsilon\epsilon_1\epsilon_2 = -1$.

Define $u_1 = z_1 z_2$, $u_2 = z_2^2$, $u_3 = z_2 z_3$. Then $K(z_1, z_2, z_3)^{\langle\lambda\rangle} = K(u_1, u_2, u_3)$ and

$$\sigma : (u_1, u_2, u_3) \mapsto (\epsilon u_1, u_2, C/u_3)$$

where $C = u_1^2 u_2(1 - u_2)/(\epsilon - u_1^2)$.

If $\epsilon = 1$, then $K(u_1, u_2, u_3)^{\langle\sigma\rangle} = K(u_1, u_2, u_3 + (C/u_3))$.

If $\epsilon = -1$, define $u_4 = u_3(1 - \sqrt{-1}u_1)/u_1$. Then we get that

$$\sigma : (u_1, u_2, u_4) \mapsto (-u_1, u_2, u_2(1 - u_2)/u_4).$$

Thus σ acts on $K(u_2)(u_1, u_4)$ as a monomial automorphism over $K(u_2)$. Hence $K(u_1, u_2, u_4)^{\langle\sigma\rangle} = K(u_2)(u_1, u_4)^{\langle\sigma\rangle}$ is rational over $K(u_2)$.

Case 3.2. $\epsilon_2 = -1$ and $\epsilon\epsilon_1\epsilon_2 = 1$.

Define $u_1 = z_1$, $u_2 = z_2^2$, $u_3 = z_2 z_3$. The details are left to the reader.

Case 3.3. $\epsilon_2 = 1$.

This case is easier and the proof is omitted.

Case 4. $G = W_6(187)$.

The action of $G = \langle\sigma, \tau, \lambda\rangle$ is given by

$$\begin{aligned} \sigma &: (x_1, x_2, x_3) \mapsto (1/x_1, x_3, x_2), \\ \tau &: (x_1, x_2, x_3) \mapsto (1/x_1, x_1/x_3, 1/(x_1 x_2)), \\ \lambda &: (x_1, x_2, x_3) \mapsto (a_1/x_1, a_2/x_2, a_3/x_3) \end{aligned}$$

where $a_1, a_2, a_3 \in K\backslash\{0\}$ (for the action of σ and τ, see $G = W_{14}(174)$ in the proof of Theorem 3.4).

Since $G \simeq C_2 \times C_2 \times C_2$, it follows that $a_1^2 = 1$, $a_2 = a_3$, and $a_2^2 = a_1$.

Define $y_1 = x_1 x_2/x_3$, $y_2 = x_2$, $y_3 = x_3$. Then we have

$$\begin{aligned} \sigma &: (y_1, y_2, y_3) \mapsto (1/y_1, y_3, y_2), \\ \tau &: (y_1, y_2, y_3) \mapsto (y_1, y_1/y_2, 1/(y_1 y_3)), \\ \lambda &: (y_1, y_2, y_3) \mapsto (\epsilon/y_1, a/y_2, a/y_3) \end{aligned}$$

where $\epsilon \in \{1,-1\}$ and $a^2 = \epsilon$.

Define

$$u = \frac{y_2 - \frac{y_1}{y_2}}{y_2y_3 - \frac{1}{y_2y_3}}, \quad v = \frac{y_3 - \frac{1}{y_1y_3}}{y_2y_3 - \frac{1}{y_2y_3}}.$$

By Theorem 2.3, we find that $K(y_1, y_2, y_3)^{\langle\tau\rangle} = K(y_1, u, v)$ and

$$\begin{aligned}\sigma &: (y_1, u, v) \mapsto (1/y_1, v, u),\\ \lambda &: (y_1, u, v) \mapsto (\epsilon/y_1, u/(ay_1), y_1v/a).\end{aligned}$$

Thus $\langle\sigma,\lambda\rangle$ acts on $K(y_1, u, v)$ by monomial K-automorphisms. By Theorem 3.4, $K(y_1, u, v)^{\langle\sigma,\lambda\rangle}$ is rational over K. □

Theorem 3.7. *Let $G = W_i(187)$ for $7 \le i \le 14$. Then $K(x_1, x_2, x_3)^G$ is rational over K.*

Proof. Case 1. $G = W_7(187)$.

The action of $G = \langle\sigma,\tau\rangle$ is given by

$$\begin{aligned}\sigma &: (x_1, x_2, x_3) \mapsto \left((\sqrt{-1})^j x_1, x_3, b/x_2\right),\\ \tau &: (x_1, x_2, x_3) \mapsto (b_1/x_1, b_2x_3, b_3x_2)\end{aligned}$$

where $b, b_1, b_2, b_3 \in K\backslash\{0\}$ and $j \in \{0,1,2,3\}$ (for the action of σ, see Case 1. $G = W_1(187)$ in the proof of Theorem 3.5).

Define $y_1 = x_1/\sqrt{b_1}$, $y_2 = x_2/\sqrt{b}$, $y_3 = b_2x_3/\sqrt{b}$. Then we have

$$\begin{aligned}\sigma &: (y_1, y_2, y_3) \mapsto \left((\sqrt{-1})^j y_1, y_3/a, a/y_2\right)\\ \tau &: (y_1, y_2, y_3) \mapsto (1/y_1, y_3, y_2)\end{aligned}$$

for some $a \in K\backslash\{0\}$ and $j \in \{0,1,2,3\}$.

Since $\tau\sigma\tau^{-1} = \sigma^{-1}$, we find that $a^2 = 1$.

For $2 \le i \le 3$, define $z_i = y_i/\sqrt{-1}$ if $a = -1$, and define $z_i = y_i$ if $a = 1$. Thus we get $K(x_1, x_2, x_3) = K(y_1, z_2, z_3)$ and

$$\begin{aligned}\sigma &: (y_1, z_2, z_3) \mapsto \left((\sqrt{-1})^j y_1, \epsilon z_3, 1/z_2\right),\\ \tau &: (y_1, z_2, z_3) \mapsto (1/y_1, z_3, z_2)\end{aligned}$$

where $\epsilon \in \{1,-1\}$ and $j \in \{0,1,2,3\}$.

Case 1.1. $\epsilon = 1$.

Define $u_i = (1 - z_i)/(1 + z_i)$ for $2 \le i \le 3$. We get

$$\begin{aligned}\sigma &: (y_1, u_2, u_3) \mapsto \left((\sqrt{-1})^j y_1, u_3, -u_2\right),\\ \tau &: (y_1, u_2, u_3) \mapsto (1/y_1, u_3, u_2).\end{aligned}$$

If $j \in \{1,3\}$, apply Theorem 2.1 and reduce the question to $K(y_1)^G$.

If $j \in \{0,2\}$, define $w_2 = u_2/u_3$, $w_3 = u_2^2$. Then $K(y_1,u_2,u_3)^{\langle\sigma^2\rangle} = K(y_1,w_2,w_3)$. Note that $\langle\sigma,\tau\rangle$ acts on $K(y_1,w_2,w_3)$ by monomial K-automorphisms and $\langle\sigma,\tau\rangle \simeq C_2 \times C_2$ as a group of automorphisms on $K(y_1,w_2,w_3)$. Thus we may apply Theorem 3.4.

Case 1.2. $\epsilon = -1$ and $j = 0$.

Define $z_1 = (1-y_1)/(1+y_1)$. Then $\sigma(z_1) = z_1$, $\tau(z_1) = -z_1$. Apply Theorem 2.1 and it suffices to consider the rationality of $K(z_2,z_3)^G$.

Case 1.3. $\epsilon = -1$ and $j = 2$.

Define

$$u = \frac{z_2 + \frac{1}{z_2}}{z_2z_3 - \frac{1}{z_2z_3}}, \quad v = \frac{z_3 + \frac{1}{z_3}}{z_2z_3 - \frac{1}{z_2z_3}}.$$

By Theorem 2.3, $K(y_1,z_2,z_3)^{\langle\sigma^2\rangle} = K(y_1,u,v)$ and

$$\begin{aligned} \sigma &: (y_1,u,v) \mapsto (-y_1, -v/(u^2-v^2), u/(u^2-v^2)) \\ \tau &: (y_1,u,v) \mapsto (1/y_1, v, u). \end{aligned}$$

Define $w_1 = u+v$, $w_2 = u-v$. Then we find that

$$\begin{aligned} \sigma &: (y_1,w_2,w_3) \mapsto (-y_1, 1/w_1, -1/w_2), \\ \tau &: (y_1,w_2,w_3) \mapsto (1/y_1, w_1, -w_2). \end{aligned}$$

We get a monomial group action. Thus $K(y_1,w_2,w_3)^{\langle\sigma,\tau\rangle}$ is rational by Theorem 3.4.

Case 1.4. $\epsilon = -1$ and $j \in \{1,3\}$. Write $\eta = (\sqrt{-1})^j$.

Define $u_2 = z_2/\eta$, $u_3 = -z_3/\eta$. We get

$$\begin{aligned} \sigma &: (y_1,u_2,u_3) \mapsto (\eta y_1, u_3, 1/u_2), \\ \tau &: (y_1,u_2,u_3) \mapsto (1/y_1, -u_3, -u_2). \end{aligned}$$

Define $v_2 = (1-u_2)/(1+u_2)$, $v_3 = (1-u_3)/(1+u_3)$. We find that $K(y_1,z_2,z_3) = K(y_1,v_2,v_3)$ and

$$\begin{aligned} \sigma &: (y_1,v_2,v_3) \mapsto (\eta y_1, v_3, -v_2), \\ \tau &: (y_1,v_2,v_3) \mapsto (1/y_1, 1/v_3, 1/v_2), \\ \sigma^2 &: (y_1,v_2,v_3) \mapsto (-y_1, -v_2, -v_3). \end{aligned}$$

Define $w_1 = y_1v_2$, $w_2 = v_2/v_3$, $w_3 = v_2^2$. Then $K(y_1,v_2,v_3)^{\langle\sigma^2\rangle} = K(w_1,w_2,w_3)$ and $\langle\sigma,\tau\rangle$ acts on $K(w_1,w_2,w_3)$ by monomial K-automorphisms because

$$\begin{aligned} \sigma &: (w_1,w_2,w_3) \mapsto (\eta w_1/w_2, -1/w_2, w_3/w_2^2), \\ \tau &: (w_1,w_2,w_3) \mapsto (-w_3/w_1, w_2, w_3). \end{aligned}$$

Thus we may apply Theorem 3.4.

Case 2. $G = W_8(187)$.

The action of $G = \langle \sigma, \tau \rangle$ is given by

$$\sigma : (x_1, x_2, x_3) \mapsto (a_1 x_1, a_2 x_3, a_3/x_2),$$
$$\tau : (x_1, x_2, x_3) \mapsto (b_1 x_1, b_2/x_3, b_3/x_2)$$

where $a_i, b_j \in K\backslash\{0\}$.

Apply Theorem 2.1 and reduce the question to $K(x_2, x_3)^G$.

Case 3. $G = W_9(187)$.

The action of $G = \langle \sigma, \tau \rangle$ is given by

$$\sigma : (x_1, x_2, x_3) \mapsto (1/x_1, b_2/x_3, b_3 x_2),$$
$$\tau : (x_1, x_2, x_3) \mapsto (c_1/x_1, c_2 x_3, c_3 x_2)$$

where $b_2, b_3, c_1, c_2, c_3 \in K\backslash\{0\}$.

Define $y_1 = x_1$, $y_2 = x_2$, $y_3 = b_2/x_3$. Use the fact that $G \simeq D_4$. We find that

$$\sigma : (y_1, y_2, y_3) \mapsto (1/y_1, y_3, a/y_2),$$
$$\tau : (y_1, y_2, y_3) \mapsto (\epsilon_1/y_1, a\epsilon_2/y_3, a\epsilon_2/y_2)$$

where $a \in K\backslash\{0\}$ and $\epsilon_1, \epsilon_2 \in \{1, -1\}$.

Define $z_i = y_i/\sqrt{a}$ for $2 \leq i \leq 3$. Then $K(x_1, x_2, x_3) = K(y_1, z_2, z_3)$ and

$$\sigma : (y_1, z_2, z_3) \mapsto (1/y_1, z_3, 1/z_2),$$
$$\tau : (y_1, z_2, z_3) \mapsto (\epsilon_1/y_1, \epsilon_2/z_3, \epsilon_2/z_2)$$

where $\epsilon_1, \epsilon_2 \in \{1, -1\}$.

Define

$$u = \frac{z_2 - \frac{1}{z_2}}{z_2 z_3 - \frac{1}{z_2 z_3}}, \quad v = \frac{z_3 - \frac{1}{z_3}}{z_2 z_3 - \frac{1}{z_2 z_3}}.$$

By Theorem 2.3, $K(y_1, z_2, z_3)^{\langle \sigma^2 \rangle} = K(y_1, u, v)$ and

$$\sigma : (y_1, u, v) \mapsto (1/y_1, -v/(u^2 - v^2), u/(u^2 - v^2)),$$
$$\tau : (y_1, u, v) \mapsto (\epsilon_1/y_1, \epsilon_2 v, \epsilon_2 u).$$

Define $w_2 = u + v$, $w_3 = u - v$. Then

$$\sigma : (y_1, w_2, w_3) \mapsto (1/y_1, 1/w_2, -1/w_3),$$
$$\tau : (y_1, w_2, w_3) \mapsto (\epsilon_1/y_1, \epsilon_2 w_2, -\epsilon_2 w_3).$$

We may apply Theorem 3.4 to assert that $K(y_1, w_2, w_3)^{\langle \sigma, \tau \rangle}$ is rational over K.

Case 4. $G = W_{10}(187)$.

The action of $G = \langle \sigma, \tau \rangle$ is given by

$$\sigma : (x_1, x_2, x_3) \mapsto (1/x_1, a_2/x_3, a_3 x_2),$$
$$\tau : (x_1, x_2, x_3) \mapsto (b_1 x_1, b_2/x_3, b_3/x_2)$$

where $a_2, a_3, b_1, b_2, b_3 \in K \backslash \{0\}$.

Define $y_1 = x_1$, $y_2 = \sqrt{a_3} x_2 / \sqrt{a_2}$, $y_3 = (\sqrt{a_2} \cdot \sqrt{a_3})/x_3$. Use the relations that $\tau^2 = 1$ and $\tau \sigma \tau^{-1} = \sigma^{-1}$. We get

$$\sigma : (y_1, y_2, y_3) \mapsto (1/y_1, y_3, 1/y_2),$$
$$\tau : (y_1, y_2, y_3) \mapsto (\epsilon_1 y_1, \epsilon_2 y_3, \epsilon_2 y_2)$$

where $\epsilon_1, \epsilon_2 \in \{1, -1\}$.

Define

$$u = \frac{y_2 - \frac{1}{y_2}}{y_2 y_3 - \frac{1}{y_2 y_3}}, \quad v = \frac{y_3 - \frac{1}{y_3}}{y_2 y_3 - \frac{1}{y_2 y_3}}.$$

By Theorem 2.3 we find that $K(y_1, y_2, y_3)^{\langle \sigma^2 \rangle} = K(y_1, u, v)$ and

$$\sigma : (y_1, u, v) \mapsto (1/y_1, -v/(u^2 - v^2), u/(u^2 - v^2)),$$
$$\tau : (y_1, u, v) \mapsto (\epsilon_1 y_1, \epsilon_2 v, \epsilon_2 u).$$

Define $z_2 = u + v$, $z_3 = u - v$. Then $\langle \sigma, \tau \rangle$ acts on $K(y_1, u, v) = K(y_1, z_2, z_3)$ by monomial K-automorphisms because

$$\sigma : (y_1, z_2, z_3) \mapsto (1/y_1, 1/z_2, -1/z_3),$$
$$\tau : (y_1, z_2, z_3) \mapsto (\epsilon_1 y_1, \epsilon_2 z_2, -\epsilon_2 z_3).$$

Thus we may apply Theorem 3.4.

Case 5. $G = W_{11}(187)$.

The action of $G = \langle \sigma, \tau \rangle$ is given by

$$\sigma : (x_1, x_2, x_3) \mapsto (a_1 x_1, x_3, a_3 x_1/x_2),$$
$$\tau : (x_1, x_2, x_3) \mapsto (1/x_1, b_2/x_3, b_3/x_2)$$

where $a_1, a_3, b_2, b_3 \in K \backslash \{0\}$.

Use the relations that $\tau^4 = \tau^2 = 1$ and $\tau \sigma \tau^{-1} = \sigma^{-1}$. We find that

$$\sigma : (x_1, x_2, x_3) \mapsto (\epsilon x_1, x_3, a_3 x_1/x_2),$$
$$\tau : (x_1, x_2, x_3) \mapsto (1/x_1, b/x_3, b/x_2)$$

where $\epsilon \in \{1, -1\}$, $a_3, b \in K \backslash \{0\}$.

Define $y_1 = x_1$, $y_2 = x_2/\sqrt{b}$, $y_3 = \sqrt{b}/x_3$ and use that relation $\tau\sigma\tau^{-1}(y_3) = \sigma^{-1}(y_3)$. We find that

$$\sigma : (y_1, y_2, y_3) \mapsto (\epsilon y_1, 1/y_3, ay_2/y_1),$$
$$\tau : (y_1, y_2, y_3) \mapsto (1/y_1, y_3, y_2)$$

where $\epsilon \in \{1, -1\}$ and $a^2 = \epsilon$.

Note that $\sigma^2 : (y_1, y_2, y_3) \mapsto (y_1, y_1/(ay_2), 1/(ay_1y_3))$.

Define

$$u = \frac{y_2 - \frac{y_1}{ay_2}}{y_2y_3 - \frac{\epsilon}{y_2y_3}}, \quad v = \frac{y_3 - \frac{1}{ay_1y_3}}{y_2y_3 - \frac{\epsilon}{y_2y_3}}.$$

By Theorem 2.3, $K(y_1, y_2, y_3)^{\langle\sigma^2\rangle} = K(y_1, u, v)$ (note that $a^2 = \epsilon$) and

$$\sigma : (y_1, u, v) \mapsto (\epsilon y_1, -y_1^2 v/(u^2 - y_1^2 v^2), u/(u^2 - y_1^2 v^2))$$
$$\tau : (y_1, u, v) \mapsto (1/y_1, v, u).$$

Case 5.1. $\epsilon = -1$.

Define $z_2 = u + y_1 v$, $z_3 = u - y_1 v$. Then $K(y_1, u, v) = K(y_1, z_2, z_3)$ and

$$\sigma : (y_1, z_2, z_3) \mapsto (-y_1, -y_1/z_3, y_1/z_2),$$
$$\tau : (y_1, z_2, z_3) \mapsto (1/y_1, z_2/y_1, -z_3/y_1).$$

Apply Theorem 3.4.

Case 5.2. $\epsilon = 1$.

Define z_2 and z_3 as in Case 1. The details are omitted.

Case 6. $G = W_{12}(187)$.

The action of $G = \langle\sigma, \tau\rangle$ is given by

$$\sigma : (x_1, x_2, x_3) \mapsto (a_1x_1, a_2x_3, a_3x_1/x_2),$$
$$\tau : (x_1, x_2, x_3) \mapsto (\epsilon x_1, x_3, x_2)$$

where $a_1, a_2, a_3 \in K\backslash\{0\}$ and $\epsilon \in \{1, -1\}$.

Since $\tau^4 = 1$ and $\tau\sigma\tau^{-1} = \sigma^{-1}$, we find that $a_1 = \epsilon$, $a_2^2 = 1$.

Define $y_1 = x_1$, $y_2 = x_2/\sqrt{a_3}$, $y_3 = x_3/\sqrt{a_3}$. We get that

$$\sigma : (y_1, y_2, y_3) \mapsto (\epsilon_1 y_1, \epsilon_2 y_3, y_1/y_2),$$
$$\tau : (y_1, y_2, y_3) \mapsto (\epsilon_1 y_1, y_3, y_2)$$

where $\epsilon_1, \epsilon_2 \in \{1, -1\}$.

Define $z_1 = y_1$, $z_2 = y_2$, $z_3 = 1/y_3$. Then we have

$$\begin{aligned}
\sigma &: (z_1, z_2, z_3) \mapsto (\epsilon_1 z_1, \epsilon_2/z_3, z_2/z_1), \\
\tau &: (z_1, z_2, z_3) \mapsto (\epsilon_1 z_1, 1/z_3, 1/z_2), \\
\sigma^2 &: (z_1, z_2, z_3) \mapsto (z_1, \epsilon_2 z_1/z_2, \epsilon_1\epsilon_2/(z_1 z_3)).
\end{aligned}$$

Define

$$u = \frac{z_2 - \frac{\epsilon_2 z_1}{z_2}}{z_2 z_3 - \frac{\epsilon_1}{z_2 z_3}}, \quad v = \frac{z_3 - \frac{\epsilon_1\epsilon_2}{z_1 z_3}}{z_2 z_3 - \frac{\epsilon_1}{z_2 z_3}}.$$

Then proceed by the same way as in the previous Case 5. $G = W_{11}(187)$. Note that $\sigma : (u, v) \mapsto (-\epsilon_1\epsilon_2 z_1^2 v/(u^2 - \epsilon_1 z_1^2 v^2),\ \epsilon_2 u/(u^2 - \epsilon_1 z_1^2 v^2))$. Thus, if $\epsilon_1 = -1$, we should take the change of variables $u \pm \sqrt{-1} z_1 v$, instead of the "usual" change of variables $u \pm z_1 v$. The details are left to the reader.

Case 7. $G = W_{13}(187)$.

The action of $G = \langle\sigma, \tau\rangle$ is given by

$$\begin{aligned}
\sigma &: (x_1, x_2, x_3) \mapsto (1/x_1, 1/x_3, a_3 x_2/x_1), \\
\tau &: (x_1, x_2, x_3) \mapsto (b_1/x_1, b_2/x_3, b_3/x_2)
\end{aligned}$$

where $a_3, b_1, b_2, b_3 \in K\backslash\{0\}$.

Use the relations $\tau^2 = 1$ and $\tau\sigma\tau^{-1} = \sigma^{-1}$. We find that $b_1 = 1$ and $b_2 = b_3 \in \{1, -1\}$. Define $y_1 = x_1$, $y_2 = \sqrt{a_3} x_2$, $y_3 = x_3/\sqrt{a_3}$. Then we get

$$\begin{aligned}
\sigma &: (y_1, y_2, y_3) \mapsto (1/y_1, 1/y_3, y_2/y_1), \\
\tau &: (y_1, y_2, y_3) \mapsto (1/y_1, \epsilon/y_3, \epsilon/y_2)
\end{aligned}$$

where $\epsilon \in \{1, -1\}$.

If $\epsilon = 1$, apply Theorem 1.2.

If $\epsilon = -1$, note that $\sigma^2 : (y_1, y_2, y_3) \mapsto (y_1, y_1/y_2, y_1/y_3)$. Define

$$u = \frac{y_2 - \frac{y_1}{y_2}}{y_2 y_3 - \frac{y_1^2}{y_2 y_3}}, \quad v = \frac{y_3 - \frac{y_1}{y_3}}{y_2 y_3 - \frac{y_1^2}{y_2 y_3}}. \tag{3.3}$$

The proof is almost the same as the previous Case 5. $G = W_{11}(187)$ by taking the change of variables $z_2 = u + v$, $z_3 = u - v$ because $\sigma : (u, v) \mapsto (-v/(u^2 - v^2), u/(u^2 - v^2))$ and $\tau : (u, v) \mapsto (-y_1 v, -y_1 u)$.

Case 8. $G = W_{14}(187)$.

The action of $G = \langle\sigma, \tau\rangle$ is given by

$$\begin{aligned}
\sigma &: (x_1, x_2, x_3) \mapsto (1/x_1, 1/x_3, a_3 x_2/x_1), \\
\tau &: (x_1, x_2, x_3) \mapsto (\epsilon x_1, b x_3, x_2/b)
\end{aligned}$$

where $a_3, b \in K\backslash\{0\}$, $\epsilon \in \{1, -1\}$.

Define $y_1 = x_1$, $y_2 = \sqrt{a_3}x_2$, $y_3 = x_3/\sqrt{a_3}$. Use the relation $\tau\sigma\tau^{-1} = \sigma^{-1}$. We get

$$\begin{aligned}
\sigma &: (y_1, y_2, y_3) \mapsto (1/y_1, 1/y_3, y_2/y_1), \\
\tau &: (y_1, y_2, y_3) \mapsto (\epsilon y_1, a y_3, y_2/a)
\end{aligned}$$

where $a^2 = \epsilon \in \{1, -1\}$.

Define u and v by Formula (3.3) in the previous Case 7. $G = W_{13}(187)$. Note that $\sigma : (y_1, u, v) \mapsto (1/y_1, -v/(u^2 - v^2), u/(u^2 - v^2))$, $\tau : (y_1, u, v) \mapsto (\epsilon y_1, av, \epsilon a u)$.

Discuss the situations $\epsilon = 1$ and $\epsilon = -1$ separately. The proof is almost the same as in the Case 7. $G = W_{13}(187)$. □

Theorem 3.8. *Let $G = W_1(194), W_2(195)$. Then $K(x_1, x_2, x_3)^G$ is rational over K.*

Proof. Case 1. $G = W_1(194)$.

The action of $G = \langle \sigma, \tau, \lambda \rangle$ is given by

$$\begin{aligned}
\sigma &: (x_1, x_2, x_3) \mapsto \left((\sqrt{-1})^j x_1, \epsilon x_3, 1/x_2\right), \\
\tau &: (x_1, x_2, x_3) \mapsto (1/x_1, x_3, x_2), \\
\lambda &: (x_1, x_2, x_3) \mapsto (a_1/x_1, a_2/x_2, a_3/x_3)
\end{aligned}$$

where $j \in \{0, 1, 2, 3\}$, $\epsilon \in \{1, -1\}$, $a_1, a_2, a_3 \in K\backslash\{0\}$ (for the actions of σ and τ, see Case 1. $G = W_7(187)$ in the proof of Theorem 3.7).

Use the relations $\sigma\lambda = \lambda\sigma$ and $\tau\lambda = \lambda\tau$. We find that $j = 0$ or 2, $a_2 = a_3 \in \{1, -1\}$. We write these actions as follows

$$\begin{aligned}
\sigma &: (x_1, x_2, x_3) \mapsto (\epsilon_1 x_1, \epsilon_2 x_3, 1/x_2), \\
\tau &: (x_1, x_2, x_3) \mapsto (1/x_1, x_3, x_2), \\
\lambda &: (x_1, x_2, x_3) \mapsto (a/x_1, \epsilon_3/x_2, \epsilon_3/x_3)
\end{aligned}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{1, -1\}$ and $a \in K\backslash\{0\}$.

Define

$$u = \frac{x_2 - \frac{\epsilon_2}{x_2}}{x_2 x_3 - \frac{1}{x_2 x_3}}, \quad v = \frac{x_3 - \frac{\epsilon_2}{x_3}}{x_2 x_3 - \frac{1}{x_2 x_3}}.$$

By Theorem 2.3 we find that $K(x_1, x_2, x_3)^{\langle \sigma^2 \rangle} = K(x_1, u, v)$ and

$$\begin{aligned}
\sigma &: (x_1, u, v) \mapsto (\epsilon_1 x_1, -v/(u^2 - v^2), u/(u^2 - v^2)), \\
\tau &: (x_1, u, v) \mapsto (1/x_1, v, u), \\
\lambda &: (x_1, u, v) \mapsto (a/x_1, \epsilon_2\epsilon_3 u, \epsilon_2\epsilon_3 v).
\end{aligned}$$

Define $y_2 = u + v$, $y_3 = u - v$. Then we get

$$\begin{aligned}
\sigma &: (y_2, y_3) \mapsto (1/y_2, -1/y_3), \\
\tau &: (y_2, y_3) \mapsto (y_2, -y_3), \\
\lambda &: (y_2, y_3) \mapsto (\epsilon_2\epsilon_3 y_2, \epsilon_2\epsilon_3 y_3).
\end{aligned}$$

Thus we may apply Theorem 3.6 and conclude that $K(x_1, y_2, y_3)^{\langle\sigma,\tau,\lambda\rangle}$ is rational over K.

Case 2. $G = W_2(195)$.
The action of $G = \langle\sigma, \tau, \lambda\rangle$ is given by

$$\begin{aligned}
\sigma &: (x_1, x_2, x_3) \mapsto (\epsilon x_1, x_3, ax_1/x_2),\\
\tau &: (x_1, x_2, x_3) \mapsto (1/x_1, b/x_3, b/x_2),\\
\lambda &: (x_1, x_2, x_3) \mapsto (c_1/x_1, c_2/x_2, c_3/x_3)
\end{aligned}$$

where $a, b, c_1, c_2, c_3 \in K\backslash\{0\}$, $\epsilon \in \{1, -1\}$ (see Case 5. $G = W_{11}(187)$ in the proof of Theorem 3.7).

Define $y_1 = x_1$, $y_2 = x_2/\sqrt{b}$, $y_3 = \sqrt{b}/x_3$ and use the relations $\tau\sigma\tau^{-1} = \sigma^{-1}$, $\sigma\lambda = \lambda\sigma$, $\tau\lambda = \lambda\tau$. We find that

$$\begin{aligned}
\sigma &: (y_1, y_2, y_3) \mapsto (\epsilon y_1, 1/y_3, ay_2/y_1),\\
\tau &: (y_1, y_2, y_3) \mapsto (1/y_1, y_3, y_2),\\
\lambda &: (y_1, y_2, y_3) \mapsto (\epsilon_1/y_1, \epsilon_2/y_2, \epsilon_2/y_3)
\end{aligned}$$

where $\epsilon, \epsilon_1, \epsilon_2 \in \{1, -1\}$ and $a^2 = \epsilon_1$.

We will "copy" the proof of Case 5. $G = W_{11}(187)$ in Theorem 3.7 without changing anything. Note that $\lambda : (u, v) \mapsto (\epsilon\epsilon_2 au/y_1, \epsilon\epsilon_2 avy_1)$.

Following the proof of $G = W_{11}(187)$, we define $z_2 = u + y_1v$, $z_3 = u - y_1v$. Since $\lambda(y_1) = \epsilon_1/y_1$, we find that $\lambda : (z_2, z_3) \mapsto (\epsilon\epsilon_2 az_2/y_1, \epsilon\epsilon_2 az_3/y_1)$ if $\epsilon_1 = 1$; while $\lambda : (z_2, z_3) \mapsto (\epsilon\epsilon_2 az_3/y_1, \epsilon\epsilon_2 az_2/y_1)$ if $\epsilon_1 = -1$.

Thus we may discuss various cases when $\epsilon, \epsilon_1 = \pm 1$ and prove that $K(y_1, z_2, z_3)^{\langle\sigma,\tau,\lambda\rangle}$ is rational over K by Theorem 3.6. $\square$

Proof of Theorem 1.3. By all the theorems proved in this section, i.e. Theorems 3.1 – 3.8, the proof of Theorem 1.3 is completed. $\square$

§4. Another proof of Theorem 1.3

In this section we will give an alternative, short proof of Theorem 1.3 in the case where the ground field K is algebraically closed and has characteristic 0. The proof is geometric and does not require computations. Moreover it does not use the classification of finite subgroups in $GL_3(\mathbb{Z})$ [Ta].

Throughout this section we assume that G is a finite 2-group and K is an algebraically closed field of characteristic 0.

The following Lemma 4.1, Lemma 4.3, Corollary 4.2 and Corollary 4.4 are meant to replace a classification of finite subgroups of $GL_2(\mathbb{Z})$ and $GL_3(\mathbb{Z})$. (In fact, finite subgroups of $GL_2(\mathbb{Z})$ are listed in [Ha1; Ha2] and those of $GL_3(\mathbb{Z})$ are listed in [Ta].)

Lemma 4.1 is an easy exercise in the representation theory of finite groups.

Lemma 4.1. *Let G be a finite 2-group. Then any representation $G \to GL_3(\mathbb{Q})$ is reducible.*

Corollary 4.2. *Let G be a finite 2-group. Then any integral representation $G \to GL_3(\mathbb{Z})$ has a subrepresentation of $\mathbb{Z}$-rank 1.*

Now let G be a finite 2-group acting monomially on $K(x, y, z)$. Let $\rho : G \to GL_3(\mathbb{Z})$ be the integral representation introduced in Definition 2.7. By Corollary 4.2 there is a 1-dimensional subrepresentation ρ_1. Denote the quotient representation by $\rho_2 : G \to GL_2(\mathbb{Z})$. In a suitable basis in $\mathbb{Z}^3$ any element of $\rho(G)$ can be written as

$$\rho(g) = \begin{pmatrix} a & b_1 & b_2 \\ 0 & c_{11} & c_{12} \\ 0 & c_{21} & c_{22} \end{pmatrix}$$

and then

$$\rho_2(g) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

We will denote by M the lattice $\mathbb{Z}^3$. Thus the group G acts on M. By Lemma 2.8 we may assume that the representation ρ is faithful on M.

Lemma 4.3. *Any 2-subgroup in $GL_2(\mathbb{Z})$ is conjugated to a subgroup of the group generated by two matrices A and B (and isomorphic to D_4)*

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Outline of the proof. For any matrix C in the group we have $\operatorname{tr} C = 0$ or ± 1 and $\det C = \pm 1$. Since C is an element of order 2^k, its minimum polynomial divides $t^{2^k} - 1$. This immediately implies $\operatorname{tr} C = 0$ and $C^4 = E$. Hence C is conjugate to either A or B, or $\pm E$. On the other hand, the order of the group is at most 8 by the famous Minkowski theorem (see, for example, [Ta, page 169]). $\square$

Let $N \subset G$ be the maximal subgroup that acts trivially on L. It is easy to see that so is the restriction $\rho_2|_N$.

Corollary 4.4. *In the above notation, there are only the following possibilities:*

$$N \simeq D_4, \quad C_4, \quad C_2, \quad or \quad N = \{1\}. \tag{4.1}$$

By Lemma 4.3, for any $g \in G$, the action of g on $K(x, y, z)$ can be written as follows:

$$g_1 : x \longmapsto \alpha_1 x^{a_1},\ y \longmapsto \Psi_1 z,\ z \longmapsto \Psi_2 y^{-1},$$
$$g_2 : x \longmapsto \alpha_2 x^{a_2},\ y \longmapsto \Psi_3 y,\ z \longmapsto \Psi_4 z^{-1},$$

where $a_i \in \{1, -1\}$, $\alpha_i \in K^\times$, and $\Psi_i \in K(x)^\times$ are some monomials, i.e. elements of the form cx^n where $c \in K^\times$ and n is an integer.

Remark. Define $L := K(x)$. We may regard $K(x, y, z)$ as the function field of the projective surface $\mathbb{P}^1_L \times \mathbb{P}^1_L$ over L and the induced action of G on $\mathbb{P}^1_L \times \mathbb{P}^1_L$ is semi-linear. Here y and z are non-homogeneous coordinates on the first and second factors respectively.

We use some facts about del Pezzo surfaces. Good references are [Ma1; Ma2; MT]. Recall that over an algebraically closed field there are two types of del Pezzo surfaces of degree 8: the Hirzebruch surface $\mathbb{F}_1$ and $\mathbb{P}^1 \times \mathbb{P}^1$. A form S of $\mathbb{F}_1$ over any field L is not minimal, that is, it contains a unique (-1)-curve. Contracting it we get a Brauer-Severi variety S' having an L-point. Hence, $S' \simeq \mathbb{P}^2_L$. A form of $\mathbb{P}^1 \times \mathbb{P}^1$ over a field L is called a *minimal del Pezzo surface of degree* 8. Taking into account the remark after Corollary 4.4, one find that Theorem 1.3 is a consequence of Proposition 4.5 below.

Proposition 4.5. *Let L be a field of characteristic* 0 *and G be a finite* 2*-group. Let S a minimal del Pezzo surface of degree* 8 *defined over L. Suppose G acts on both L and S which is regarded as an L^G-scheme, and let N be the kernel of the homomorphism $G \to \operatorname{Aut} L$. Assume that N is given as in* (4.1)*. If L^G is a C_1-field, then S/G is rational over L^G.*

For the definition C_i-fields, see [Gr, page 3; DKT, page 1256]. An algebraic function field of one variable over an algebraically closed field is a C_1-field by Tsen's Theorem [Gr, page 22; DKT, Theorem 5.8, page 1259].

Proof. Our proof is by induction on the order of G.

If $G = \{1\}$, then S has an L-point (because L is a C_1- field) and so S is a rational quadric in $\mathbb{P}^3$.

From now on we assume that $n := |G| > 1$ and the assertion holds for all groups G_1 with $|G_1| < n$.

Let $\bar{L}$ be the algebraic closure. Set $\bar{S} := S \otimes \bar{L}$. The group N acts linearly on S over L and on $\bar{S}$ over $\bar{L}$. In particular, N acts on $\operatorname{Pic}(\bar{S}) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

First we assume that $|N| > 2$. By Formula (4.1) there is an element $g \in N$ of order 4 and $\tau := g^2$ is contained in $Z(G) \cap N$, where $Z(G)$ is the center of G. Indeed, in this case, τ is the only element of order 2 in $Z(N) \supset Z(G) \cap N$. Denote $N_0 := \{1, \tau\}$ and $F := S/N_0$. By our assumptions, F is defined over L and G/N_0 acts naturally on F by semi-linear automorphisms. Thus $S/G \simeq F/(G/N_0)$. In a suitable (non-homogeneous) coordinate system in $\bar{F} \simeq \mathbb{P}^1_{\bar{L}} \times \mathbb{P}^1_{\bar{L}}$ over $\bar{L}$ there are only the following possibilities for the action:

(1) $\tau : x \longmapsto y, \quad y \longmapsto x;$

(2) $\tau : x \longmapsto x, \quad y \longmapsto -y;$

(3) $\tau : x \longmapsto -x, \quad y \longmapsto -y.$

The element $\tau = g^2$ acts trivially on $\operatorname{Pic}(\bar{S}) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Thus the case (1) is impossible. In the case (2) we have $\bar{F} \simeq \mathbb{P}^1_{\bar{L}} \times \mathbb{P}^1_{\bar{L}}$, so F a minimal del Pezzo surface of degree 8. We reduce the problem to an action of a smaller group $G_1 = G/N$ on F.

Consider the case (3). Let $\pi : S \to F$ be the quotient morphism. Over the field $\bar{L}$, the group N acts freely in codimension one on S and has exactly 4 fixed points. These points give us 4 singular points on F of type A_1 and F has no other singularities. In particular, this implies that the divisor K_F is Cartier. Since the map $\pi : S \to F$ is étale in codimension one, $K_S = \pi^* K_F$. Hence $-K_F$ is ample and $K_F^2 = K_S^2/2 = 4$. This means that F is a del Pezzo surface of degree 4 having 4 singular points of type A_1. According to the classification [CT], there are exactly 4 lines $l_1, \dots, l_4$ on F. Let $\mu : \tilde{F} \to F$ be the minimal resolution and let $\tilde{l}_i$ be the proper transform of l_i. Then $\tilde{F}$ is a *weak del Pezzo* surface in the sense that its anti-canonical divisor $-K_{\tilde{F}}$ is nef and big (such surfaces are also called generalized del Pezzo surfaces [CT]). Moreover, $K_{\tilde{F}}^2 = K_F^2 = 4$. Again by [CT, Prop. 6.1] $\tilde{l}_1, \dots, \tilde{l}_4$ are disjointed (-1)-curves (and $\tilde{F}$ contains no other (-1)-curves). Consider their contraction $\varphi : \tilde{F} \to F'$. Here $K_{F'}^2 = K_{\tilde{F}}^2 + 4 = 8$. Further, $-K_{F'} = \varphi_*(-K_{\tilde{F}})$ is nef and big, i.e., F' is a weak del Pezzo surface of degree 8. Moreover, every (-2)-curve C on $\tilde{F}$ meets two φ-exceptional curves $\tilde{l}_i$. Hence the surface F' contains no curves with negative self-intersection numbers. Therefore, F' is a minimal del Pezzo surface of degree 8. Finally, both the birational maps $\mu : \tilde{F} \to F'$ and $\varphi : \tilde{F} \to F$ are canonically defined, so they are G/N-equivariant. Again we reduce the problem to an action of $G_1 = G/N$ on F' with $|G_1| < |G|$.

Now assume that $|N| = 2$, so $N = \{1, \tau\}$. As above, $S/G \simeq F/(G/N_0)$ and in a suitable coordinate system in $\bar{F} \simeq \mathbb{P}^1_{\bar{L}} \times \mathbb{P}^1_{\bar{L}}$ over $\bar{L}$ there are only the possibilities (1), (2) and (3) listed above. In cases (2) and (3) we may argue as above. It remains to consider the case (1). Then the quotient $\bar{F} = \bar{S}/N$ is isomorphic to $\mathbb{P}^2_{\bar{L}}$. Thus F is a del Pezzo surface of degree 9 (a Brauer-Severi variety). In this situation, $G_1 = G/N$ acts on L effectively. Then by Lemma 4.6 below the quotient $S/G \simeq \mathbb{P}^2_L/(G/N)$ is L-rational.

Finally, if $N = \{1\}$, i.e., the action of G on L is effective, we may also apply Lemma 4.6 to finish the proof. $\square$

Lemma 4.6. *Let $\Bbbk$ be a field of characteristic 0 and let S be a del Pezzo surface of degree $d \geq 5$ over $\Bbbk$. Suppose a finite group G acts effectively on both $\Bbbk$ and S which is regarded as a $\Bbbk^G$-scheme. If $\Bbbk^G$ is a C_1-field, then S/G is rational over $\Bbbk^G$.*

Proof. We claim that the natural G-equivariant morphism $S \to S/G \times_{\Bbbk^G} \Bbbk$ is an isomorphism. Since it is a local property, it suffices to show that, if G acts on an affine domain R over a field $\Bbbk$ so that $\Bbbk$ is left invariant by the action of G and G acts faithfully both on $\Bbbk$ and R, then $R^G \otimes_{\Bbbk^G} \Bbbk \simeq R$. This follows from the Galois descent lemma [KMRT, Lemma 18.1, page 279]. Note that the Galois descent lemma is essentially another form of Theorem 2.1.

Now consider the following diagram

$$\begin{array}{ccc} S & \longrightarrow & S/G \\ \downarrow & & \downarrow \\ \operatorname{Spec} \Bbbk & \to & \operatorname{Spec} \Bbbk^G \end{array}$$

Since $S/G \times_{\Bbbk^G} \Bbbk \simeq S$, it follows that S/G is a minimal del Pezzo surface of degree $d \geq 5$. By [Ma2, Chapter 4, §7] it is sufficient to show that S/G has a $\Bbbk^G$-point. This holds for an arbitrary $\Bbbk$ if $d = 5$ [SD] and for C_1-fields if $d \geq 6$ [Ma1]. Thus S/G is $\Bbbk^G$-rational. $\square$

§5. Applications

Proof of Theorem 1.4. By 2.2, we may assume that V contains no 1-dimensional subrepresentation. By [Se, pages 53 and 65], every irreducible representation of G is an induced representation from some 1-dimensional representation of H for some subgroup H of G; thus it is a monomial representation. It follows that V is isomorphic to (i) an irreducible 4-dimensional representation, (ii) the direct sum of two irreducible 2-dimensional representations, or (iii) the direct sum of three 2-dimensional irreducible representations.

Case 1. V is an irreducible 4-dimensional representation.

Since V is induced from a 1-dimensional representation, for any $\sigma \in G$, $\sigma(e_i) = a_i(\sigma) \cdot e_j$ for some basis e_1, e_2, e_3, e_4 of V where $a_i(\sigma) \in K \backslash \{0\}$. If x_1, x_2, x_3, x_4 denotes the dual basis of e_1, e_2, e_3, e_4, then $K(V) = K(x_1, x_2, x_3, x_4)$ and $\sigma \cdot x_i = b_i(\sigma) \cdot x_j$ for any $\sigma \in G$ where $b_i(\sigma) \in K \backslash \{0\}$.

Define $y_i = x_i/x_4$ for $1 \leq i \leq 3$. Then $\sigma(x_4) = t_\sigma \cdot x_4$ where $t_\sigma \in K(y_1, y_2, y_3)$ for any $\sigma \in G$. Apply Theorem 2.2. We find that $K(x_1, x_2, x_3, x_4)^G = K(y_1, y_2, y_3)^G(f)$ for some f satisfying $\sigma(f) = f$ for any $\sigma \in G$. Note that G acts on $K(y_1, y_2, y_3)$ by monomial K-automorphisms. Apply Theorem 1.3. We find that $K(y_1, y_2, y_3)^G$ is rational over K. Thus the quotient $P(V)/G$ is also rational.

Case 2. V is a direct sum of irreducible 2-dimensional representations and certain 1-dimensional representations.

Apply Theorem 2.2 and Theorem 1.1. The details are omitted.

Case 3. $V \simeq V_1 \oplus V_2 \oplus V_3$ where each V_i is the representation space of an irreducible 2-dimensional representation of G. It follows that $K(V) = K(x_1, x_2, y_1, y_2, z_1, z_2)$ and G acts on elements of the set $\{x_1, x_2\}$ (resp. $\{y_1, y_2\}$, $\{z_1, z_2\}$) by permutations up to non-zero elements in K. Define $x = x_1/x_2, y = y_1/y_2, z = z_1/z_2$. Apply Theorem 2.2 repeatedly. We reduce the question to the rationality of $K(x, y, z)^G$. Apply Theorem 1.3. $\square$

For the proof of Theorem 1.5, we need two standard results in group theory, whose proof will be omitted.

Lemma 5.1. *Let p be a prime number, G be a finite p-group. Denote by $Z(G)$ the center of G. Let H be a normal subgroup of G. If $H \supsetneq \{1\}$, then $H \cap Z(G) \supsetneq \{1\}$.*

Lemma 5.2. *Let G be a finite 2-group of exponent e, and K be a field containing a primitive e-th root of unity. For each non-negative integer i, let s_i be the number of non-equivalent 2^i-dimensional irreducible representations of G over K. Then $\sum_i s_i 4^i = |G|$.*

Proof of Theorem 1.5 when K is algebraically closed. Let G be a group of order 32. If G is abelian, then $K(G)$ is rational by Fischer's Theorem [Sw, Theorem 6.1]. Thus we will assume that G is non-abelian from now on. Let $Z(G)$ be the center of G, and $[G, G]$ be the commutator subgroup of G.

If $\operatorname{char} K = 2$, then $K(G)$ is rational by Kuniyoshi's Theorem [Ku]. Thus we will assume that $\operatorname{char} K \neq 2$ and K is algebraically closed from now on.

Case 1. $|Z(G)| \geq 8$.

Since $|G/Z(G)| \leq 4$, $G/Z(G)$ is isomorphic to $C_2 \times C_2$. Thus there is an abelian subgroup H so that $Z(G) \subset H \subset G$ and $[G : H] = 2$. Apply Theorem 2.6. We find that $K(G)$ is rational.

Case 2. $Z(G) \simeq C_2$ or C_4.

Let $\tau \in Z(G)$ be the unique element of order 2 in $Z(G)$.

We claim that there is an irreducible representation $\rho : G \to GL(V)$ over K, which is faithful.

Assume the validity of this claim. Note that $\dim_K V \leq 4$ by Lemma 5.2. Hence $K(V)^G$ is rational over K by Theorem 1.4. Since $K(G)$ is rational over $K(V)^G$ by Theorem 2.1, we find that $K(G)$ is rational.

Now we will prove this claim. Suppose not, i.e. every irreducible representation $\rho : G \to GL(W)$ is not faithful. Then $\operatorname{Ker}(\rho) \cap Z(G) \supsetneq \{1\}$ by Lemma 5.1. Thus $\tau \in \operatorname{Ker}(\rho) \cap Z(G)$. We conclude that $\rho(\tau)$ is trivial for all irreducible representations of G. But the regular representation of G is faithful and is the direct sum of these irreducible representations of G. This leads to a contradiction.

Case 3. $Z(G) \simeq C_2 \times C_2$.

We will show that G has a faithful 4-dimensional representation, which is either irreducible or a direct sum of two 2-dimensional irreducible representations. In any case, the rationality of $K(G)$ can be proved as in Case 2.

Step 1. If G has a faithful 4-dimensional representation, we are done. From now on, we will assume that, if G has a 4-dimensional irreducible representation, then it is not faithful.

We will prove that G has no 4-dimensional irreducible representation at all.

Suppose that $\rho : G \to GL_4(K)$ is irreducible. Then $\operatorname{Ker}(\rho) \supsetneq \{1\}$. Thus ρ gives rise to a faithful irreducible 4-dimensional representation $G/\operatorname{Ker}(\rho) \to GL_4(K)$. Note that $|G/\operatorname{Ker}(\rho)| \leq 16$ which is impossible by Lemma 5.2.

Step 2. We will show that G has at least four non-equivalent 2-dimensional irreducible representation.

Let G has p 1-dimensional irreducible representations and has s 2-dimensional irreducible representations. Then $p + 4s = |G| = 32$ and $p = |G/[G,G]|$ which is a divisor of 16. The only solutions of (p,s) are $(p,s) = (4,7)$, $(8,6)$, $(16,4)$. Thus $s \geq 4$.

Step 3. From Step 2, let $\rho_i : G \to GL_2(K)$ be all the 2-dimensional irreducible representations where $1 \leq i \leq s$ with $s \geq 4$.

We claim that, there is some ρ_i so that $\mathrm{Ker}(\rho_i) \not\supset Z(G)$.

Otherwise, assume that $\mathrm{Ker}(\rho_i) \supset Z(G)$ for all $1 \leq i \leq s$. Note that the kernel of any 1-dimensional representation contains $[G,G]$. Hence $[G,G] \cap (\bigcap_{1\leq i\leq s} \mathrm{Ker}(\rho_i)) \supset [G,G] \cap Z(G) \supsetneq \{1\}$ by Lemma 5.1. It follows that there is some element $g \in G$, $g \neq 1$, but g belongs to the kernel of any irreducible representation of G. This is impossible by the same arguments as in the last paragraph of Case 2.

Step 4. By Step 3, we may assume that $\mathrm{Ker}(\rho_1) \not\supset Z(G)$. Write $\mathrm{Ker}(\rho_1) \cap Z(G) = \langle\lambda\rangle$ where λ is an element of order two in $Z(G) \simeq C_2 \times C_2$.

Define $H = \bigcap_{1\leq i\leq s} \mathrm{Ker}(\rho_i)$. We will prove that $H = \{1\}$.

Suppose that $H \supsetneq \{1\}$. But Lemma 5.1, $H \cap Z(G) \supsetneq \{1\}$. But $H \cap Z(G) \subset \mathrm{Ker}(\rho_1) \cap Z(G) = \langle\lambda\rangle$. Hence $H \cap Z(G) = \langle\lambda\rangle$. Thus the group $G/\langle\lambda\rangle$ has at least four 2-dimensional irreducible representations. If the number of 1-dimensional irreducible representations of $G/\langle\lambda\rangle$ is q, then $q + 4 \cdot 4 \leq |G/\langle\lambda\rangle|$ by Lemma 5.2. Since $|G/\langle\lambda\rangle| = 16$, we get a contradiction.

Step 5. Since $H = \{1\}$, we find that $\lambda \notin H$. We may assume that $\lambda \notin \mathrm{Ker}(\rho_2)$.

Define $H_0 = \mathrm{Ker}(\rho_1) \cap \mathrm{Ker}(\rho_2)$. If $H_0 \supsetneq \{1\}$, then $H_0 \cap Z(G) \supsetneq \{1\}$ by Lemma 5.1. Hence $H_0 \cap Z(G) \subset \mathrm{Ker}(\rho_1) \cap Z(G) = \langle\lambda\rangle$. Thus $H_0 \cap Z(G) = \langle\lambda\rangle$, which is impossible because $\lambda \notin \mathrm{Ker}(\rho_2)$.

Step 6. Since $\{1\} = H_0 = \mathrm{Ker}(\rho_1) \cap \mathrm{Ker}(\rho_2)$, the direct sum of ρ_1 and ρ_2 is a faithful representation of G. Done. $\square$

Remark. As we note in Section 1, the above proof is valid only under the assumption that $\sqrt{a} \in K$ for any $a \in K$ instead of the weaker assumption that $\zeta_e \in K$ (which is given in Theorem 1.5). Consider the following proposition : *Let G be a non-abelian group of order 32 and let e be the exponent of G. If K is a field containing ζ_e and $G \to GL(V)$ is an irreducible representation over K where $dimV = 4$, then $K(V)^G$ is rational over K.* Note that the above proposition is valid if we assume the classification of groups of order 32, because there are exactly seven such groups (they are groups $(42;49)$, $(43;50)$, $(44;43)$, $(45;44)$, $(46;6)$, $(47;7)$, $(48;8)$ in the notation of [CHKP, page 3025]) and all the 4-dimensional irreducible representations can be described explicitly. On the other hand, at present we don't have a proof of the above proposition if we try to avoid using the classification of groups of order 32. With this proposition in hand, the above proof of Theorem 1.5 may be adapted to the situation assuming only that $\zeta_e \in K$.

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